Extending perfect matchings to Hamiltonian cycles in line graphs

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joint work with John Baptist Gauci, Domenico Labbate, Giuseppe Mazzuoccolo and Jean Paul Zerafa

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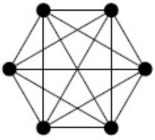

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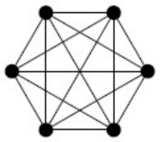

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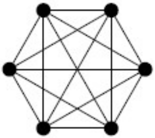

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and that a *perfect matching* or 1-*factor* of a graph G
is a 1-regular spanning subgraph, i.e.
a subgraph consisting of independent edges and containing all vertices of G.

Example

K_6 *and one of its perfect matchings*

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A graph G is *Hamiltonian* if it contains a spanning cycle

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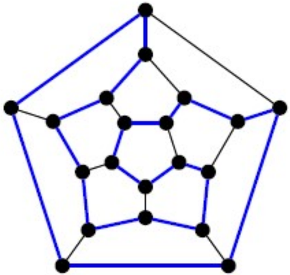

Hamiltonian cycle in the Dodecahedron

Further definitions

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- A *clique* in a graph G is a complete subgraph of G, and so K_n may sometimes be referred to as an n-*clique*.

Perfect Matching Hamiltonian

Let G be a graph admitting a perfect matching. If for every perfect matching M of G there exists another perfect matching N such that $M \cup N$ is a hamiltonian cycle, we say that G is *Perfect Matching Hamiltonian* or PMH, for short.

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Graphs with this property have been studied since the $1970s$ by Las Vergnas [14] and Häggkvist [10] where they were called F-hamiltonian.

Sufficient *Ore type* conditions for PMH

Theorem (Las Vergnas 1972 [14])

Let $G = (U, V)$ be a bipartite graph, with $|U| = |V| = \frac{n}{2} \geqslant 2$. If for each pair of non-adjacent vertices $u \in U$ and $v \in V$ we have $deg(u) + deg(v) \geqslant \frac{n}{2} + 2$, then G is PMH.

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Theorem (Häggkvist 1979 [10])

Let G be a graph, with $|V(G)| = n \geqslant 4$ and n even. If for each pair of non-adjacent vertices u and v we have $deg(u) + deg(v) \geqslant n + 1$, then G is PMH.

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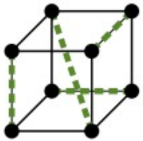
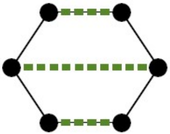


Pairing of Q_3



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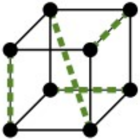

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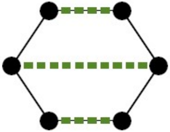

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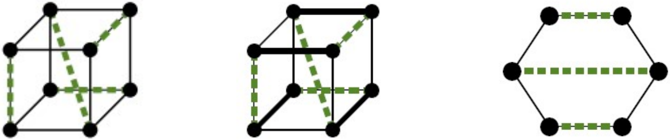

Pairing of Q_3 *Extended pairing of Q_3* *Pairing of C_6*

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Not always. The one in the cube Q_3 can, but the one in C_6 can't.

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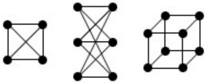
Theorem (Alahmadi, Aldred, Alkenani, Hijazi, Solé and Thomassen 2015 [3])

Let G be a pairing hamiltonian cubic graph. Then G is:



K_4 $K_{3,3}$ or Q_3

Cartesian Product

The Cartesian product $G\square H$ of two graphs G and H

$$V(G\square H) = \{(u_r, v_s) : u_r \in V(G) \text{ and } v_s \in V(H)\}; \text{ and}$$

$$E(G\square H) = \{(u_i, v_j)(u_k, v_l) : u_i = u_k, v_jv_l \in E(H) \text{ or } u_iu_k \in E(G), v_j = v_l\}$$

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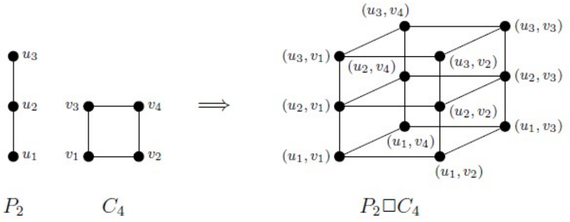

Hypercubes

Definition

A d-dimensional hypercube Q_d is the d-fold Cartesian product of P_2s, i.e $P2\square P2\square\cdots\square P2$, with P_2 appearing d times in the product.

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Theorem (Fink - 2007 [7])

For every perfect matching P of K_{Q_d} there exists a perfect matching R of Q_d, $d \geqslant 2$, such that $P \cup R$ is a Hamilton cycle of K_{Q_d}. So, Q_d is PH, hence also PMH.

More on Hypercubes

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Conjecture (Ruskey and Savage - 1993 [17])

Every matching (not necessarily perfect) of Q_d extends to a hamilton cycle.

Proved to be true for $d = 2, 3, 4$ (Fink 2007 [7]) and for $d = 5$ (Wang and Zhao 2018 [18]).

More results

Further results on PMH-graphs were found by Yang 1999 [19] .

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In 2009 [4] , Amar, Flandrin and Gancarzewicz gave a degree sum condition on three independent vertices of a graph, for it to be PMH.

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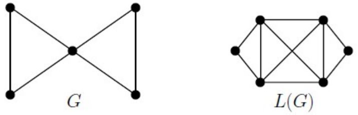

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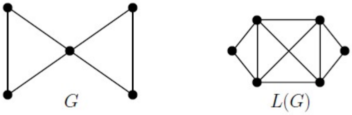

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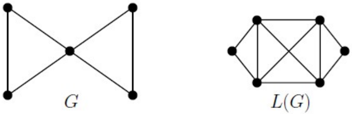

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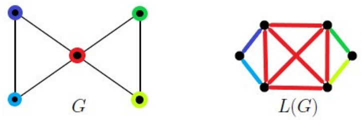

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- We say that it is the *canonical clique partition* of $L(G)$.

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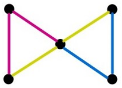

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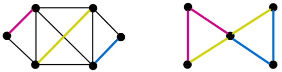

Corollary (Sumner - 1965 [16])

G connected with even size $\Longrightarrow L(G)$ has a perfect matching.

There is a natural bijection between the paths in a P_3-decomposition of G and the edges of the corresponding perfect matching M of $L(G)$.

Hamiltonicity of Line graphs

Theorem (Harary and Nash-Williams - 1965 [11])

$L(G)$ is Hamiltonian if and only if G admits a dominating tour.

In particular, this implies that if G is Hamiltonian or Eulerian, then, $L(G)$ is also Hamiltonian, but the converse is not necessarily true.

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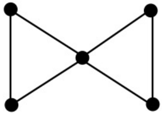

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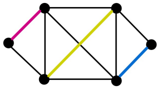

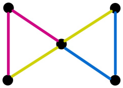

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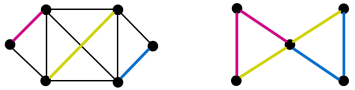

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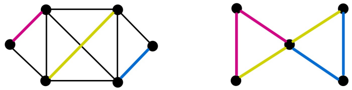

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- The sequence of edges in this Euler tour corresponds to a sequence of vertices in $L(G)$ which gives a Hamiltonian cycle H of $L(G)$

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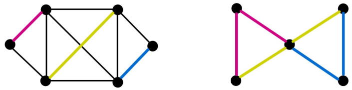

- Since G is arbitrarily traceable from some vertex $\Longrightarrow \exists$ an Euler tour in which every pair of edges in the same 3-path are consecutive.
- The sequence of edges in this Euler tour corresponds to a sequence of vertices in $L(G)$ which gives a Hamiltonian cycle H of $L(G)$
- two edges of each 3-path in the P_3-decomposition are consecutive in the Euler tour, H contains all the edges of M, as required. □

Complete graphs

In 1976 Daykin proved

Theorem (Daykin - 1976 [5])

If the edges of the complete graph K_n, for $n \geqslant 6$, are coloured in such a way that no three edges of the same colour are incident to any given vertex, then there exists a properly coloured Hamiltonian cycle.

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Which we have used to prove that

Theorem (M.A., J.B.G., D.L., G.M., J.P.Z. - 2021 [1])

For $n \equiv 0, 1 \mod 4$, $L(K_n)$ *is PMH.*

Complete graphs have PMH line graphs

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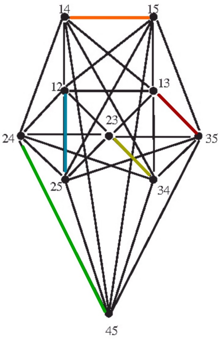

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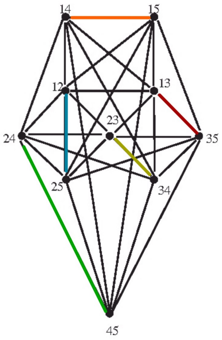

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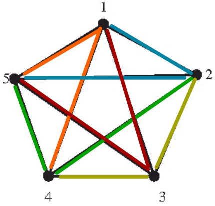

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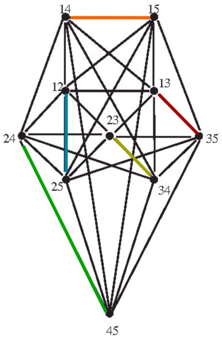

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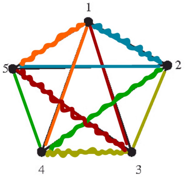

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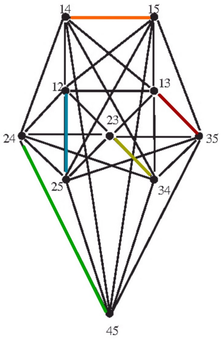

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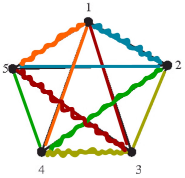

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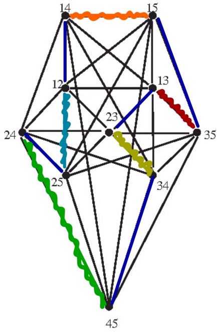

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Complete Bipartite graphs

Theorem (C.C. Chen and D.E. Daykin - 1976 [6])

Consider an edge-colouring of the complete bipartite graph $K_{m,m}$ such that no vertex is incident to more than k edges of the same colour. If $m \geqslant 25k$, then there exists a properly coloured Hamiltonian cycle.

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So for the case $k = 2$, one could obtain that $L(K_{m,m})$ is PMH for every even $m \geqslant 50$.

However, using a different and more technical approach we proved

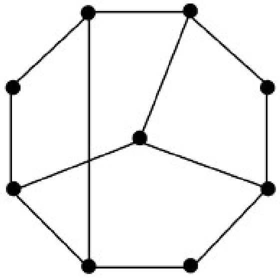
Theorem (M.A., J.B. Gauci, J.P. Zerafa - 2021 [2])

Let m_1 be an even integer and let $m_2 \geqslant 1$. Then, $L(K_{m_1,m_2})$ does not have the PH–property if and only if $m_1 = 2$ and m_2 is odd.

Technical Lemma

Lemma (M.A., J.B. Gauci, D. Labbate, G. Mazzuoccolo, J.P. Zerafa - 2021 [1])

Let G be a connected graph such that $\Delta(G) \leqslant 3$. A perfect matching M of $L(G)$ can be extended to a Hamiltonian cycle if and only if there exists a dominating cycle D of G such that the vertices in G untouched by D correspond to a subset of cliques in Ω not intersected by M, where Ω is the canonical clique partition of $L(G)$.



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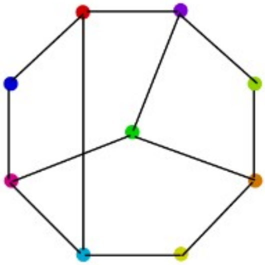

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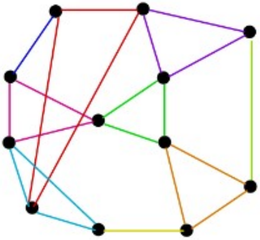

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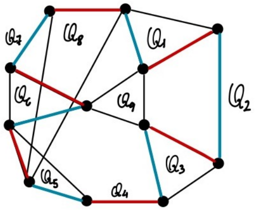

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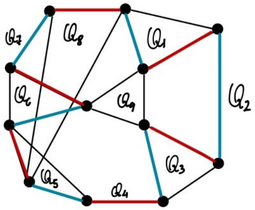

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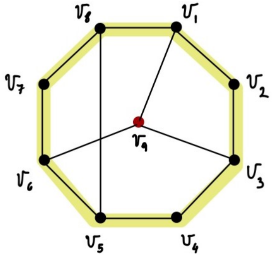

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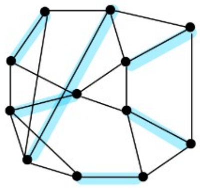

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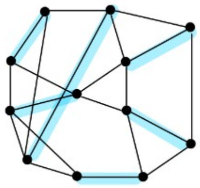

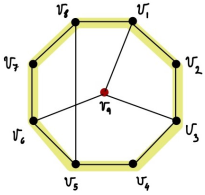

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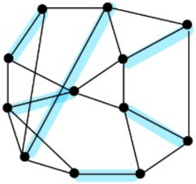
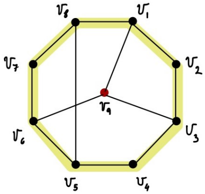
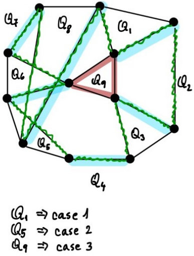
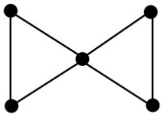






False for $\Delta(G) > 3$



Has no dominating cycle but its line graph is PMH

Hamiltonian subcubic

Theorem (M.A., J.B. Gauci, D. Labbate, G. Mazzuoccolo, J.P. Zerafa - 2021 [1])

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M can be extended to a Hamiltonian cycle of $L(G)$. □

Cubic case

Theorem (Kotzig - 1964 [12])

A cubic graph G is Hamiltonian if and only if $L(G)$ partitions into two Hamiltonian cycles.

Corollary (M.A., J.B. Gauci, D. Labbate, G. Mazzuoccolo, J.P. Zerafa - 2021 [1])

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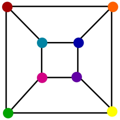

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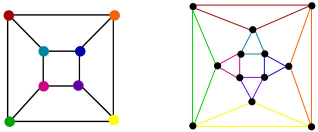

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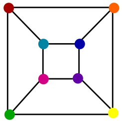

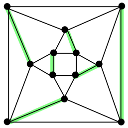

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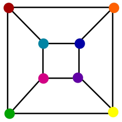

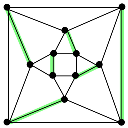

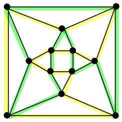

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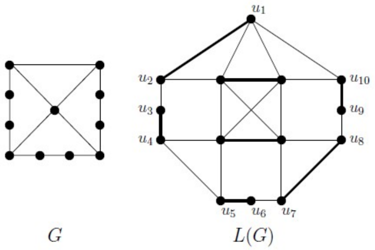

A Hamiltonian graph with maximum degree 4 whose line graph is not PMH.

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In G there is a dominating cycle (of length $|V(G)| - 1$) avoiding v, from which M extends to a Hamiltonian cycle. □

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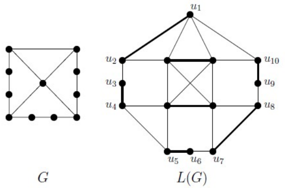
OpenProblems

Recall that



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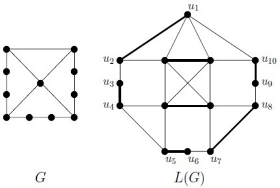
Problem

Let G be an r-regular Hamiltonian graph of even size, for $r \geqslant 4$. Does $L(G)$ have the PMH-property?

OpenProblems

Recall that



A Hamiltonian graph with maximum degree 4 whose line graph is not PMH.

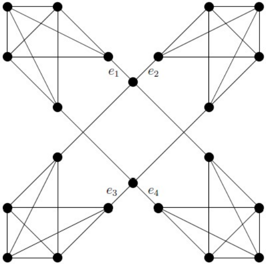
Problem

Let G be an r-regular Hamiltonian graph of even size, for $r \geqslant 4$. Does $L(G)$ have the PMH-property?

Progress in this direction has recently been made by Gauci and Zerafa [8] and [9] for a family of quartic graphs and for the cartesian product of two cycles.

OpenProblems

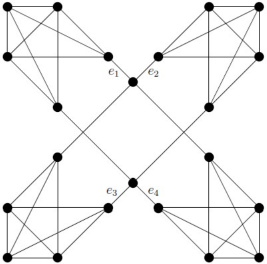
Eulerian non-Hamiltonian 4-regular graph
whose line graph is not PMH.



Every perfect matching of $L(G)$ which contains the edges e_1e_2 and e_3e_4 cannot be extended to a Hamiltonian cycle of $L(G)$.

OpenProblems

Eulerian non-Hamiltonian 4-regular graph
whose line graph is not *PMH*.



Every perfect matching of $L(G)$ which contains the edges e_1e_2 and e_3e_4 cannot be extended to a Hamiltonian cycle of $L(G)$.

Problem

Let G be a graph of even size which is both Eulerian and Hamiltonian. Does $L(G)$ have the PMH-property?

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Thanks for your attention!