# Which finite groups are filled?

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Open University Discrete Mathematics Seminar

June 8, 2022

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# Abstract

A non-empty subset $S$ of a group $G$ is called a product-free set if $S$ and $SS$ have no element in common. Let $S$ be a maximal by inclusion product-free set in a finite group $G$. We say that $S$ fills $G$ if every non-identity element of $G$ is contained in the union of $S$ and $SS$. A finite group $G$ is called a filled group if every maximal by inclusion product-free set in $G$ fills $G$. In this talk, we shall give an application of product-free sets to combinatorics, as well as discuss the known finite filled groups.

# Outline of this presentation

# Father of Sum-free sets: Issai Schur (1875-1941)



**Theorem (Schur 1917)**

*For any partition of the positive integers into a finite number of parts, one of the parts contains three integers $x, y$ and $z$ with $x + y = z$.*

# On a paper of Cameron and Erdős



## Definition

Let $S$ be a subset of $\mathbb{Z}$. Then $S$ is **sum-free** if the equation $x = y + z$ does not hold for $x, y, z \in S$.

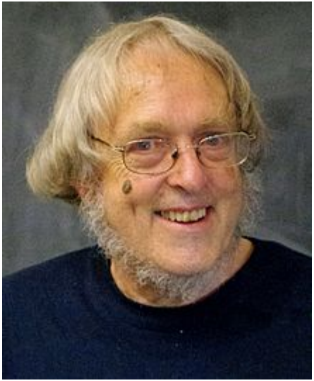
## Observation (Cameron, Erdős 1988)

Let $S$ be a subset of $\{1, \cdots, n\}$. Then $|S| \leq \lceil \frac{n}{2} \rceil$. Moreover, the only sum-free sets of size $\lceil \frac{n}{2} \rceil$ are the following:
(i) the complete list of odd numbers in $\{1, \cdots, n\}$;
(ii) $\{\frac{n+1}{2}, \cdots, n\}$ if $n$ is odd, and $\{\frac{n}{2}, \cdots, n-1\}$ and $\{\frac{n}{2}+1, \cdots, n\}$ if $n$ is even.

As the number of odd numbers in $\{1, \cdots, n\}$ is $\lceil \frac{n}{2} \rceil$, the number of subsets of odd numbers in $\{1, \cdots, n\}$ is $2^{\frac{n}{2}}$.

# Cameron-Erdős Conjecture





(a) Peter Cameron

(b) Paul Erdős

**Conjecture (Cameron-Erdős '1988)**

The number of sum-free sets in $\{1, \cdots, n\}$ is $O(2^{\frac{n}{2}})$.

# Cameron-Erdős Conjecture Proved!

(a) Ben Green

(b) Alexander Sapozhenko

There have been many trials to prove the famous Cameron-Erdős conjecture. However, in 2003, Ben Green and Alexander Sapozhenko independently proved it.

# Product-free sets

## Definition

Let $G$ be a finite group, and $S$ a non-empty subset of $G$. Then $S$ is **product-free** if the equation $xy = z$ has no solution in $S$. Equivalently, if $S \cap SS = \varnothing$, where $SS = \{xy | \; x, y \in S\}$.

- Let $H$ be a subgroup of a group $G$. Then any non-trivial coset of $H$ is product-free.
- Let $N$ be a normal subgroup of $G$, and $\omega : G \longrightarrow G/N$ be the canonical homomorphism. If $Q$ is product-free in $G/N$, then $\omega^{-1}(Q)$ is product-free in $G$.
- A product-free set in a finite group $G$ has size at most $\frac{|G|}{2}$.

A maximal by cardinality product-free set is called a **maximal product-free set**. We denote the cardinality of a maximal product-free set (MPFS for short) in a finite group $G$ by $\lambda(G)$.

# On maximal product-free sets



**Theorem (Erdős 1965)**

*Let $G$ be a finite abelian group. Then* $\frac{2}{7}|G| \leq \lambda(G) \leq \frac{|G|}{2}$.

# Product-free sets: Part I (MPFS) Contd.





### 1. *Maximal Sum-Free Sets of Elements of Finite Groups*

By Palahenedi Hewage DIANANDA and Hian Poh YAP
Department of Mathematics, University of Singapore, Singapore

(Comm. by Zyoiti SUETUNA, M. J. A., Jan. 13, 1969)

**1.** *Introduction.* Let $G$ be an additive group. If $S$ and $T$ are non-empty subsets of $G$, we write $S\pm T$ for $\{s\pm t\,;\, s\in S,\ t\in T\}$ respectively, $|S|$ for the cardinal of $S$ and $\bar{S}$ for the complement of $S$ in $G$. We abbreviate $\{f\}$, where $f\in G$ to $f$. We say that $S$ is sum-free in $G$ if $S$ and $S+S$ have no common element and that $S$ is maximal sum-free in $G$ if $S$ is sum-free in $G$ and $|S|\geqq|T|$ for every $T$ sum-free in $G$. We denote by $\lambda(G)$ the cardinal of a maximal sum-free set in $G$. We say that $S$ is in arithmetic progression with the difference $d$ if $S=\{s,\, s+d,\, s+2d,\, \cdots,\, s+nd\}$ for some $s$ and $d\in G$ and some integer $n\geqq 0$.

In [3] Yap obtained certain results concerning $\lambda(G)$ for abelian $G$. The main purpose of this paper is to generalize and to improve, where possible, his results.

**2.** *Abelian groups.* Throughout this section $G$ is an abelian group. We use the following theorem [2 ; p. 6] due to M. Kneser :

**Theorem 1.** *Let $A$ and $B$ be finite non-empty subsets of $G$. Then a subgroup $H$ of $G$ exists such that $A+B+H=A+B$ and $|A+B|\geqq|A+H|+|B+H|-|H|$.*

Suppose that $S$ is a maximal sum-free set in $G$. Then a subgroup $H$ of $G$ exists such that

$$S+S+H=S+S \quad \text{and} \quad |S+S|\geqq 2|S+H|-|H|. \qquad (1)$$

**Lemma 1.** *$S+H$ is also a sum-free set in $G$.*

**Proof.** Otherwise, $S+H$ and $(S+H)+(S+H)=S+S$ have a common element. Thus $s+h=s_1+s_2$ for some $s$, $s_1$ and $s_2\in S$ and some $h\in H$. Hence $s=s_1+s_2-h\in S+S+H=S+S$. This is not possible since $S$ is sum-free in $G$.

# Sizes of MPFS in finite abelian groups



## Definition

Let $G$ be a finite abelian group. We say $G$ is of **type I** if $|G|$ is divisible by a prime $p \equiv 2 \bmod 3$. On the other hand, $G$ is of **type II** if 3 is a factor of $|G|$, and $|G|$ has no prime factor $\equiv 2$ mod 3. Finally, $G$ is of **type III** if every prime factor of $|G|$ is a prime $p \equiv 1$ *mod* 3.

## Theorem (Diananda and Yap 1969)

*Let $G$ be a finite abelian group.*
*(i) If $G$ is of type I, then $\lambda(G) = \frac{|G|}{3}(1 + \frac{1}{p})$, where $p$ is the least prime factor of $|G|$ such that $p \equiv 2$ mod 3.*
*(ii) If $G$ is of type II, then $\lambda(G) = \frac{|G|}{3}$.*
*(iii) If $G$ is of type III, then $\frac{|G|}{3}(1 - \frac{1}{m}) \leq \lambda(G) \leq \frac{1}{3}(|G| - 1)$, where $m$ is the exponent of $G$.*

# Green and Ruzsa '05



(a) Ben Green

(b) Imre Z. Ruzsa

The size of a maximal free-free set in any finite abelian group is known. Thanks to Green and Ruzsa.

# Babai and Sós

- The question on what is the size of a maximal product-free set in a finite group has been attributed to Babai and Sós who raised the following question:

**Question (Babai and Sós '1985)**

Does there exists a constant $c > 0$ such that every group of order $n$ has a product-free set of size greater than $cn$?

- Combining the result given here for finite abelian groups with the lifting argument, a bound on the sizes of maximal product-free sets in finite soluble groups is decidable, and subsequently a lower bound on the density of their maximal product-free sets.
- To completely answer this question of what is the size of a maximal product-free set in a finite group, we only need to consider the finite simple groups.

# CFSG



- The classification of finite simple groups (CFSG) shows that every non-cyclic finite simple group is isomorphic to one of the following:

  (i) Alternating group of degree $n$ ($n \geq 5$);

  (ii) a simple group of Lie type defined over $\mathbb{F}_q$ (of classical or exceptional type);

  (iii) one of the 26 sporadic groups.

- Using this classification, one can show that if $G$ is a finite non-cyclic simple group, then there exists a constant $c > 0$ such that $G$ has a proper subgroup of index at most $c|G|^{3/7}$; so $\lambda(G) \geq \frac{1}{c}|G|^{4/7}$.

# Kedlaya (1998)



- Kedlaya improved this bound by showing that there exists a constant $c > 0$ such that $\lambda(G) > c|G|^{11/14}$; in particular he showed that if a largest subgroup $H$ of $G$ has index $m$, then one can find a union of $c\sqrt{m}$ non-trivial cosets of $H$ that is product-free.
- He then asked a weaker version of the first question of Babai and Sós: given $\epsilon > 0$, does there exists a product-free set of size greater than $c_\epsilon|G|^{1-\epsilon}$?
- Gowers answered this question in the negative by showing that for sufficiently large $q$, the group $PSL_2(q)$ has no product-free set of size $c|PSL_2(q)|^{8/9}$.
- The property of $PSL_2(q)$ he used is that it has no non-trivial irreducible representation of low dimension.

# Gowers (2008)





- Gowers gave several equivalence of quasirandomness. He called a group quasirandom if it contains no non-trivial irreducible representation of low dimension.
- Clearly, no finite abelian group is quasirandom since any irreducible representation of such group has dimension one.
- Gowers showed that if a group contains no large product-free set, then it is quasirandom.
- He showed that if $G$ has a non-trivial representation of dimension $d$, then there exists a constant $k > 0$ such that $G$ has a product-free set of size at least $k^d|G|$.

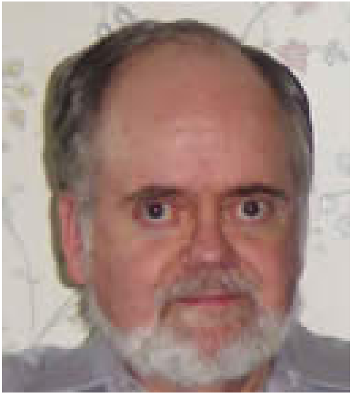
**Theorem (Gowers '2008)**

*If the smallest non-trivial representation of $G$ is of dimension $d$, then $\lambda(G) \leq d^{-1/3}|G|$.*

# Founders of 'locally maximal product-free sets'



(a) Anne Penfold Street



(b) Earl G. Whitehead Jr.

# Founding paper on locally maximal product-free sets



JOURNAL OF COMBINATORIAL THEORY (A) 17, 219–226 (1974)

## Group Ramsey Theory

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*Communicated by the Managing Editors*

Received April 16, 1973

A subset $S$ of a group $G$ is said to be a *sum-free set* if $S \cap (S+S) = \varnothing$. Such a set is *maximal* if for every sum-free set $T \subseteq G$, we have $|T| \leqslant |S|$. Here, we generalize this concept, defining a sum-free set $S$ to be *locally maximal* if for every sum free set $T$ such that $S \subseteq T \subseteq G$, we have $S = T$. Properties of locally maximal sum-free sets are studied and the sets are determined (up to isomorphism) for groups of small order.

# Locally maximal product-free sets (LMPFS)



## Definition

A **locally maximal product-free set** (LMSFS for short) is a product-free subset $S$ of $G$ such that given any other product-free subset $T$ of $G$ with $S \subseteq T$, then $S = T$.

## Lemma (A characterisation of LMPFS)

Suppose $S$ is a product-free set in the group $G$. Then $S$ is locally maximal product-free if and only if $G = T(S) \cup \sqrt{S}$, where $T(S) = S \cup SS \cup S^{-1}S \cup SS^{-1}$ and $\sqrt{S} = \{x \in G | x^2 \in S\}$.

## Remark

Every maximal product-free set is locally maximal, but the converse is not necessarily true. In the latter case, $\{x, x^4, x^7\}$ is locally maximal in $C_8$, but not maximal.

# Groups containing small LMPFS

(a) Michael Giudici

(b) Sarah Hart

**Question**

For a fixed positive integer $k$, which finite groups contain locally maximal product-free set of size $k$?

# Groups containing LMPFS of sizes 1 and 2



## Proposition (Giudici, Hart '09)

Let $S$ be a locally maximal product-free set in $G$. Then $\langle S \rangle$ is a normal subgroup of $G$. In addition, $G/\langle S \rangle$ is either trivial or an elementary abelian 2-group.

## Theorem

*If a finite group $G$ contains a LMPFS of size 1, then $G$ is one of $C_2$, $C_3$, $C_4$ or $Q_8$.*

## Theorem (Giudici, Hart' 2009)

*If a finite group $G$ contains a LMPFS of size 2, then $G$ is one of the following: $C_4$, $C_5$, $C_6$, $D_6$, $C_7$, $C_8$, $C_2^2$, $C_4 \times C_2$, $C_2 \times Q_8$, $Q_{12}$, $C_2 \times Q_8$ or $\langle g, h |\; g^4 = 1 = h^4, hg = g^{-1}h \rangle$.*

# On finite groups containing LMPFS of size 3



Giudici and Hart proved that $|G| \leq 4|S|^2 + 1$ when $S \cap S^{-1} = \varnothing$, and determined all groups of order up to 37 that contain LMPFS of size 3. The largest group size appearing there is 24.

**Theorem (Giudici, Hart' 2009)**

*Suppose $S$ is a locally maximal product-free set of size 3 in a group $G$ such that not every two element subset of $S$ generates $\langle S \rangle$. Then $|G| \leq 24$.*

Finally, they concluded the paper with the following conjecture:

**Conjecture (Giudici, Hart '09)**

If a finite group $G$ contains a locally maximal sum-free set of size 3, then $|G| \leq 24$.

# On the proof of the Conjecture of Giudici and Hart



**Theorem (A, Hart 2005)**

*Suppose $S$ is a LMPFS of size 3 in a group $G$ such that every two element subset of $S$ generates $\langle S\rangle$. Then $|G| \leq 24$.*

**Corollary (A, Hart 2005)**

*If a group $G$ contains a LMPFS $S$ of size 3, then $|G| \leq 24$ and the only possibilities for $G$ and $S$ are listed in the Table at https://www.hindawi.com/journals/ijcom/2016/8939182/.*

# Groups containing LMPFS of size 3

| $G$ | | $S$ |
|---|---|---|
| $\langle g : g^6 = 1 \rangle$ | $\cong C_6$ | $\{g, g^3, g^5\}$ |
| $\langle g, h : g^3 = h^2 = 1, hgh = g^{-1} \rangle$ | $\cong D_6$ | $\{h, gh, g^2h\}$ |
| $\langle g : g^8 = 1 \rangle$ | $\cong C_8$ | $\{g, g^{-1}, g^4\}$ |
| $\langle g, h : g^4 = h^2 = 1, hgh^{-1} = g^{-1} \rangle$ | $\cong D_8$ | $\{h, gh, g^2\}$ |
| $\langle g : g^9 = 1 \rangle$ | $\cong C_9$ | $\{g, g^3, g^8\}, \{g, g^4, g^7\}$ |
| $\langle g, h : g^3 = h^3 = 1, gh = hg \rangle$ | $\cong C_3 \times C_3$ | $\{g, h, g^2h^2\}$ |
| $\langle g : g^{10} = 1 \rangle$ | $\cong C_{10}$ | $\{g^2, g^5, g^8\}, \{g, g^5, g^8\}$ |
| $\langle g : g^{11} = 1 \rangle$ | $\cong C_{11}$ | $\{g, g^3, g^5\}$ |
| $\langle g : g^{12} = 1 \rangle$ | $\cong C_{12}$ | $\{g^2, g^6, g^{10}\}$ |
| | | $\{g, g^6, g^{10}\}, \{g, g^3, g^8\}$ |
| $\langle g, h : g^6 = 1, g^3 = h^2, hgh^{-1} = g^{-1} \rangle$ | $\cong Q_{12}$ | $\{g, g^3, g^5\}$ |
| Alternating group of degree 4 | $= \mathrm{Alt}(4)$ | $\{x, y, z : x^2 = y^2 = z^3 = 1\}$ |
| | | $\{x, z, xzx : x^2 = z^3 = 1\}$ |
| | | $\{x, z, zxz : x^2 = z^3 = 1\}$ |
| $\langle g : g^{13} = 1 \rangle$ | $\cong C_{13}$ | $\{g, g^3, g^9\}, \{g, g^6, g^{10}\}$ |
| $\langle g : g^{15} = 1 \rangle$ | $\cong C_{15}$ | $\{g, g^3, g^{11}\}$ |
| $\langle g, h : g^4 = h^4 = 1, gh = hg \rangle$ | $\cong C_4 \times C_4$ | $\{g, h, g^{-1}h^{-1}\}$ |
| $\langle g, h : g^8 = 1, g^4 = h^2, hgh^{-1} = g^{-1} \rangle$ | $\cong Q_{16}$ | $\{g, g^4, g^{-1}\}$ |
| $\langle g, h : g^8 = h^2 = 1, hgh^{-1} = g^5 \rangle$ | (order 16) | $\{g, g^6, g^3h\}$ |
| $\langle g, h : g^{10} = 1, g^5 = h^2, hgh^{-1} = g^{-1} \rangle$ | $\cong Q_{20}$ | $\{g, g^5, g^8\}, \{g^2, g^5, g^8\}$ |
| $\langle g, h : g^3 = h^7 = 1, ghg^{-1} = h^2 \rangle$ | $\cong C_7 \rtimes C_3$ | $\{gh, gh^{-1}, g^{-1}\}$ |
| $\langle x : x^3 = 1 \rangle \times \langle g, h : g^4 = 1, g^2 = h^2, hgh^{-1} = g^{-1} \rangle$ | $\cong C_3 \times Q_8$ | $\{g^2, xg^2, x^2g^2\}$ |
| $\langle g, h : g^{12} = 1, g^6 = h^2, hgh^{-1} = g^{-1} \rangle$ | $\cong Q_{24}$ | $\{g^2, g^6, g^{10}\}$ |
| | | $\{g, g^6, g^{10}\}$ |

# Some of the ingredients of the proof

## Lemma

If $S$ is a LMPFS in a group $G$, then $S$ is locally maximal product-free in $\langle S \rangle$.

## Proposition

Suppose $S$ is a LMPFS of size 3 in $G$. If $\langle S \rangle$ is either cyclic or dihedral, then $|G| \leq 24$.

Note that the bound on $|G|$ in the above Proposition is attainable. For example, in

$$Q_{24} = \langle x, y \mid x^{12} = 1, x^6 = y^2, xy = yx^{-1} \rangle,$$

the set

$$S = \{x, x^6, x^{10}\}$$

is a LMPFS with the property that $\langle S \rangle \cong C_{12}$.

# A sketch of the proof of theorem



**Theorem.** If $S$ is a LMPFS of size 3 in a group $G$ such that every two element subset of $S$ generates $\langle S\rangle$, then $|G| \leq 24$.

- Suppose $S$ is a LMPFS of size 3 in a group $G$ such that every two element subset of $S$ generates $\langle S\rangle$.
- Clearly, $S$ contains exactly one of: (a) no involution; (b) exactly one involution; (c) at least two involutions.
- Suppose $S$ contains no involution. Then either $S \cap S^{-1} = \varnothing$ or $S \cap S^{-1} \neq \varnothing$. For the former, we use $|G| \leq 4|S|^2 - 2|S| + 1$ and we are done. For the latter, we have that $\langle S\rangle$ is cyclic and we are done.
- If $S$ contains at least two involutions, then as the group generated by any two involutions is dihedral, we have that $\langle S\rangle$ is dihedral, and we are done.
- The left over case is that $S$ contains exactly one involution. We use combinatorial properties of groups and LMPFS to resolve this case. □

# Largest $G$ containing LMPFS of small sizes

- Experimental results suggest that the largest size of a finite group containing a LMPFS of size 1, 2, 3, 4, 5 or 6 would be 8, 16, 24, 40, 64 or 96 respectively.
- Below is a list of finite groups $G$ of expected highest possible size which contain locally maximal product-free sets of size $k$ for $k \in \{1, 2, 3, 4, 5\}$.

| $k$ | $G$ |
|---|---|
| 1 | $G_8 := \langle x, y \mid x^4 = 1, x^2 = y^2, xy = yx^{-1} \rangle \cong Q_8$ |
| 2 | $G_{16A} := \langle x, y \mid x^4 = 1 = y^4, xy = y^{-1}x \rangle$ |
| | $G_{16B} := \langle x, y, z \mid x^4 = 1 = z^2, x^2 = y^2, xy = yx^{-1}, yz = zy, xz = zx \rangle \cong Q_8 \times C_2$ |
| 3 | $G_{24A} := \langle x, y \mid x^{12} = 1, x^6 = y^2, xy = yx^{-1} \rangle \cong Q_{24}$ |
| | $G_{24B} := \langle x, y, z \mid x^4 = 1 = z^3, x^2 = y^2, xy = yx^{-1}, yz = zy, xz = zx \rangle \cong Q_8 \times C_3$ |
| 4 | $G_{40A} := \langle x, y \mid x^8 = 1 = y^5, xy = y^{-1}x \rangle$ |
| | $G_{40B} := \langle x, y \mid x^{20} = 1, x^{10} = y^2, xy = yx^{-1} \rangle \cong Q_{40}$ |
| | $G_{40C} := \langle x, y, z \mid x^{10} = 1 = z^2, x^5 = y^2, xy = yx^{-1}, yz = zy, xz = zx \rangle \cong Q_{20} \times C_2$ |
| 5 | $G_{64A} := \langle a, b \mid a^8 = b^4 = b^{-2}a^{-1}b^{-2}a = ab^{-1}a^2ba = a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}ab = 1 \rangle$ |
| | $G_{64B} := \langle a, b \mid a^4 = b^8 = a^2b^{-1}a^2b = a^{-1}b^2ab^2 = (aba^{-1}b)^2 = 1 \rangle$ |
| | $G_{64C} := \langle a, b, c \mid a^4 = b^4 = c^2 = cb^{-1}cb = ca^{-1}ca = a^2b^{-1}a^{-2}b = ba^{-1}b^2ab = b^{-1}a^{-1}bab^{-1}aba^{-1} = (a^{-1}b^{-2}a^{-1})^2 = a^{-1}(ba)^2ba^{-1}b = 1 \rangle$ |
| | $G_{64D} := \langle a, b, c \mid a^4 = b^4 = c^2 = cb^{-1}cb = a^{-1}bab = bab^{-2}a^{-1}b = (aca)^2 = a^2b^{-1}a^2b = (ca^{-1})^4 = (b^{-1}a^{-2}b^{-1})^2 = (caca^{-1})^2 = 1 \rangle$ |
| | $G_{64E} := \langle a, b, c \mid a^4 = b^4 = c^2 = ba^{-1}ba = b^2cb^{-2}c = a^2ba^2b^{-1} = ca^{-1}b^{-2}ca^{-1} = (cb^{-1})^2(cb)^2 = (a^{-1}b^2a^{-1})^2 = (cbacba^{-1})^2 = 1 \rangle$ |

| k | G |
|---|---|
| 5 | $G_{64F} := \langle a, b, c \mid a^4 = b^8 = c^2 = cbcb^{-1} = ca^{-1}ca = a^{-1}bab = a^2b^{-1}a^2b = 1\rangle$ |
| | $G_{64G} := \langle x, y, z \mid x^8 = 1 = z^4, x^4 = y^2, xy = yx^{-1}, yz = zy, xz = zx\rangle \cong Q_{16} \times C_4$ |
| | $G_{64H} := \langle a, b, c \mid a^4 = c^{-1}b^{-1}cb = c^4a^2 = c^2ac^2a^{-1} = b^4a^2 = b^2ab^2a^{-1} = b^{-1}ac^2ba^{-1} = c^{-1}a^{-1}b^{-1}cb^{-1}a^{-1} = 1\rangle$ |
| | $G_{64I} := \langle a, b, c \mid a^4 = c^4 = a^{-1}bab = a^{-1}cac = c^{-1}b^{-1}cb = a^{-1}b^{-4}a^{-1} = a^2c^{-1}a^2c = (c^{-1}a^{-2}c^{-1})^2 = 1\rangle$ |
| | $G_{64J} := \langle a, b, c \mid b^4 = c^8 = ab^{-1}a^{-1}b^{-1} = c^{-1}b^{-1}cb = a^{-1}cac = b^{-1}a^{-1}ba^{-1} = 1\rangle$ |
| | $G_{64K} := \langle a, b, c, d \mid a^4 = b^4 = c^2 = d^2 = da^{-1}da = a^{-1}bab = ca^{-1}ca = db^{-1}db = (dc)^2 = cb^{-1}cb = a^2b^{-1}a^2b = (b^{-1}a^{-2}b^{-1})^2 = 1\rangle \cong G_{16A} \times C_2^2$ |
| | $G_{64L} := \langle a, b, c, d \mid a^4 = c^2 = d^2 = aba^{-1}b = db^{-1}db = (dc)^2 = cb^{-1}cb = ba^2b = da^{-1}da = b^{-1}a^{-1}ba^{-1} = a^2ca^{-2}c = (ca)^4 = cacdaca^{-1}cda^{-1} = 1\rangle$ |
| | $G_{64M} := \langle a, b, c, d \mid a^4 = b^2 = d^2 = c^2a^2 = caca^{-1} = c^2a^{-2} = (db)^2 = dc^{-1}dc = (aba)^2 = a^{-1}ba^{-1}cbc = (a^2d)^2 = da^{-1}bdab = (ba^{-1})^4 = c^{-1}bcbaba^{-1}b = 1\rangle$ |
| | $G_{64N} := \langle a, b, c, d, g \mid a^4 = c^2 = d^2 = g^2 = b^2a^2 = baba^{-1} = gb^{-1}gb = ca^{-1}ca = ga^{-1}ga = (gc)^2 = cb^{-1}cb = da^{-1}da = (gd)^2 = (dc)^2 = db^{-1}db = 1\rangle \cong Q_8 \times C_2^3$. |

# Bertram's insight from 1983





In connection with expanders of graph, Bertram, in 1983, remarked that there is ample evidence that every locally maximal sum-free set $S$ in an abelian group of even order satisfies $|\sqrt{S}| \leq 2|S|$. To better understand LMSFS, Bertram asked the questions below, which have now been answered in both abelian and nonabelian cases.

## Question (1)

Does every locally maximal sum-free set $S$ in a finite abelian group satisfy $|\sqrt{S}| \leq 2|S|$?

## Question (2)

Does there exist a sequence of finite abelian groups $G$ and locally maximal sum-free sets $S \subset G$ such that $\frac{|SS|}{|S|} \to \infty$ as $|G| \to \infty$?

## Question (3)

Does there exist a sequence of finite abelian groups $G$ and locally maximal sum-free sets $S \subset G$ such that $|S| < c|G|^{\frac{1}{2}}$ as $|G| \to \infty$, where $c$ is a constant?

# Application to Combinatorics

**Definition**

The *Ramsey number* $R_n(3)$ is the smallest positive integer such that colouring the edges of a complete graph on $R_n(3)$ vertices in $n$ colours forces the appearance of a monochromatic triangle.

- Clearly, $R_1(3) = 3$.

A symmetric product-free set (SPFS for short) is a product-free set $S$ such that $S = S^{-1}$.

**Theorem (A lower bound on Ramsey numbers)**

*If $G$ is a finite group such that $G^*$ can be partitioned into disjoint union of $m$ symmetric product-free sets (where $m \geq 2$), then $R_m(3) \geq |G| + 1$. [Note: $G^* = G \setminus \{1\}$.]*

# Proof of Theorem



- Suppose $G^* = S_1 \sqcup \cdots \sqcup S_m$ is a disjoint union of $m$ SPFS.
- We assign to the set $S_i$ colour $C_i$ for each $i \in \{1, \cdots, m\}$.
- Let $K_{|G|}$ be the complete graph on $|G|$ vertices: $v_1, v_2, \cdots, v_{|G|}$. [Vertices of $K_{|G|}$ are the elements of $G$.]
- We $m$-colour $K_{|G|}$ as follows: colour edge $v_iv_j$ with colour $C_k$ if $v_iv_j^{-1} \in S_k$. Since $S_k$ is symmetric, this induces a well-defined edge-colouring of the graph.
- Let $v_a$, $v_b$ and $v_c$ be any three vertices of $K_{|G|}$ and consider the triangle on these vertices.
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$$(v_av_b^{-1})(v_bv_c^{-1}) = v_av_c^{-1} \notin S_k.$$
  - So $v_av_c$ must be coloured $C_l$ for $l \neq k$, and no monochromatic triangle is formed.
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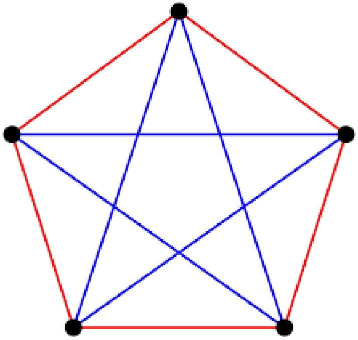
What is $R_2(3)$?

# $R_2(3) \geq 6$

M. I: $C_5^* = \{x, x^4\} \cup \{x^2, x^3\}$; so $R_2(3) > 5$.

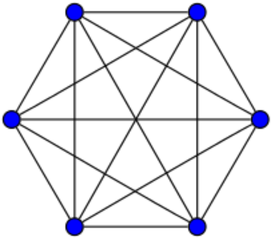
M. II: Colour the outside edges of $K_5$ with red colour, and the inside edges with blue colour.



In either case, $R_2(3) \geq 6$.

# $R_2(3) \leq 6$

- Suppose we 2-colour the edges of $K_6$ with colours blue and green.
- We label the vertices of $K_6$ as $v_0, v_1, v_2, v_3, v_4$ and $v_5$.
- Choose a vertex (say $v_0$) of $K_6$. By the Pigeonhole principle, at least three edges incident with $v_0$ must be coloured with the same colour (say blue). W.L.O.G, let those edges be $v_0v_1$, $v_0v_2$ and $v_0v_3$.
- If any of the edges $v_1v_2$, $v_1v_3$ or $v_2v_3$ is coloured blue, then we have a blue triangle. So suppose none of the three edges is coloured blue, then each of them is coloured green and we obtain a green triangle.



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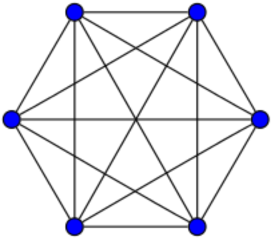

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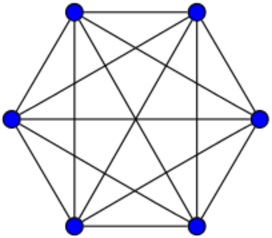

# $R_2(3) \leq 6$



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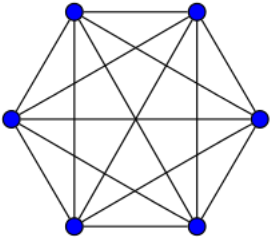

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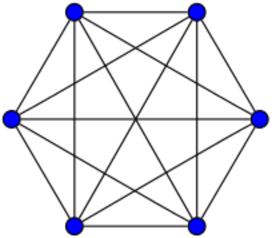

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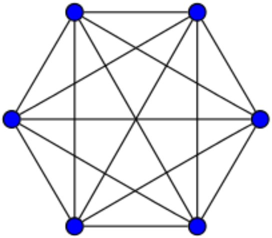

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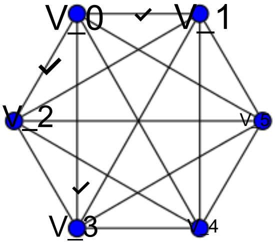


Thus $R_2(3) = 6$.

# An upper bound on Ramsey numbers

**Theorem (Greenwood, Gleason '1955)**

*For $n \geq 2$, $R_{n+1}(3) \leq (n+1)(R_n(3)-1)+2$.*

**$R_3(3) = 17$**

- From the result of Greenwood and Gleason, $R_3(3) \leq 17$.
- $C_2^4 = \langle x_1, x_2, x_3, x_4 | \ x_ix_j = x_jx_i, \ x_i^2 = 1 \text{ for } 1 \leq i,j \leq 4\rangle$. Clearly, ${C_2^4}^* =$ $\{x_1, x_2, x_3, x_4, x_1x_2x_3x_4\} \cup \{x_1x_2, x_1x_3, x_2x_4, x_1x_2x_3, x_1x_2x_4\} \cup \{x_1x_4, x_2x_3, x_3x_4, x_1x_3x_4, x_2x_3x_4\}$. In the light of the lower bound proved earlier therefore, $R_3(3) = 17$.

**Question**

What is $R_4(3)$?

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# On bounds for $R_m(3)$, where $4 \leq m \leq 7$.

- On bounds for $R_4(3)$: $51 \leq R_4(3) \leq 62$
  - 1967: Folkman ($R_4(3) \leq 65$);
  - 1973: Whitehead ($R_4(3) \geq 50$);
  - Chung: $R_{n+1}(3) \geq 3(R_n(3) - 1) + R_{n-2}(3)$ for $n \geq 3$.
  - 1995: Sánchez ($R_4(3) \leq 64$);
  - 2006 Preprint: Kramer ($R_4(3) \leq 62$).
- Known bounds on $R_5(3)$, $R_6(3)$ and $R_7(3)$
  - $162 \leq R_5(3) \leq 307$;
  - Conjecture: $257 \leq R_5(3) \leq 307$;
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# Application to finite geometry



- The problem of obtaining minimal sizes of locally maximal sum-free sets in finite groups was first raised by Street and Whitehead in 1974, and subsequently by Babai and Sós in 1985.
- This problem is also of great interest to finite geometers who study the packing problem: determination of minimal size of a complete cap in $\mathrm{PG}(n-1,2)$.
- The projective space of dimension $n$ over $\mathrm{GF}(q)$ is denoted by $\mathrm{PG}(n,q)$. A $k$-**cap** in $\mathrm{PG}(n,q)$ is a set of $k$ points, no three of which are collinear.
- A $k$-cap is called complete if it is not contained in a $(k+1)$-cap of the same projective space. **Complete caps in $\mathrm{PG}(n-1,2)$ are synonymous to locally maximal sum-free sets in $C_2^n$.**

# On filled groups

We say a product-free subset $S$ of a group $G$ *fills* $G$ if $G^* \subseteq S \sqcup SS$ (where $G^* = G \setminus \{1\}$), and $G$ is called a *filled group* if every locally maximal product-free set in $G$ fills $G$.

**Lemma (Street, Whitehead' 1974)**

Let $G$ be a finite group and $N$ a normal subgroup of $G$.
If $Q$ is a LMPFS in $G/N$ that does not fill $G/N$, then the set $R$ defined as $R := \{g \in G : gN \in Q\}$ is a LMPFS in $G$ that does not fill $G$.
That is, if $G$ is filled then $G/N$ is filled.

**Theorem (Street, Whitehead' 1974)**

*A finite abelian group is filled if and only if it is $C_3, C_5$ or an elementary abelian 2-group.*

**Lemma (Street, Whitehead' 1974)**

If $G$ is a filled group with a normal subgroup of index 3, then $G \cong C_3$.

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# Filled $p$-groups, for odd prime $p$

## Proposition (A', Hart 2015)

Suppose $G$ is a finite $p$-group, where $p$ is an odd prime. Then $G$ is filled if and only if $G$ is either $C_3$ or $C_5$.

## Proof.

- Certainly $C_3$ and $C_5$ are filled.
- For the reverse implication, let $G$ be a finite $p$-group of order $p^n$. We proceed by induction on $n$.
- If $G$ is nonabelian, then the quotient of $G/Z(G)$ is a strictly smaller $p$-group; so inductively, it is either $C_3$ or $C_5$. But it is a basic result that if $G/Z(G)$ is cyclic, then $G$ is abelian, giving a contradiction.
- Therefore $G$ is abelian, and the result follows immediately from the classification of filled abelian groups.

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# The 1974 Conjecture of Street and Whitehead

Street and Whitehead verified that all finite dihedral groups of order up to 12 are filled. They asserted that the dihedral group of order $2n$ is not a filled group for $n = 6k + 1$ $(k \geq 1)$, and went further to give the following locally maximal product-free set

$$S := \{x^{2k+1}, \ldots, x^{4k}, x^{2k+1}y, \ldots, x^{4k}y\}$$

which they claim does not fill $D_{2n}$.

**Observation**

Let $G$ be a dihedral group of order $2n$ for $n = 6k + 1$ and $k \geq 1$. Then the set $S := \{x^{2k+1}, \ldots, x^{4k}, x^{2k+1}y, \ldots, x^{4k}y\}$ is product-free but not locally maximal in $G$. In particular, $V := \{x^{2k+1}, \ldots, x^{4k}, x^{2k}y, x^{2k+1}y, \ldots, x^{4k}y\}$, which properly contains $S$, is product-free in $G$.

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It turns out that $D_{14}$ is a filled group.

# The filled groups known as at 2015



| Order | Filled groups |
|---|---|
| 2 | $C_2$ |
| 3 | $C_3$ |
| 4 | $C_2 \times C_2$ |
| 5 | $C_5$ |
| 6 | $D_6$ |
| 8 | $C_2^3,\ D_8$ |
| 10 | $D_{10}$ |
| 12 | $D_{12}$ |
| 14 | $D_{14}$ |
| 16 | $C_2^4,\ D_8 \times C_2$ |
| 22 | $D_{22}$ |
| 32 | $C_2^5,\ D_8 * Q_8.$ |

Table: Filled groups of order at most 32

# On filled dihedral groups

**Theorem (A, Hart '2015)**

*1. If a dihedral group $G$ contains a LMPFS $S$, then $|G| \leq |S|^2 + |S|$.*

*2. If $S$ is a locally maximal product-free set of size $k \geq 3$ in a finite dihedral group $G$ of order $k(k+1)$, then $S$ does not fill $G$.*

- An example of the construction above exists in $D_{20}$ as $S = \{x, x^8, y, x^5y\}$ is locally maximal in $D_{20}$ but it does not fill $D_{20}$.
- All dihedral group of order less than 16 are filled group.
- The first example of a non-filled dihedral group is $D_{16}$; a non-filling set is $\{x, x^6, y, x^4y\}$.

**Theorem (A, Erskine, Hart '2018)**

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# On filled nilpotent group

**Theorem (A, Hart' 2015)**

*The only filled groups of odd order are $C_3$ and $C_5$.*

**Lemma (A, Hart' 2015)**

If $G$ is a filled nilpotent group, then $G$ is $C_3$, $C_5$ or a 2-group.

**Theorem (A, Erskine, Hart '2018)**

*Let $G$ be a 2-group. Then $G$ is filled if and only if $G$ is either elementary abelian, or one of $D_8$, $D_8 \times C_2$, $D_8 * Q_8$ or $(D_8 * Q_8) \times C_2$.*

**Corollary**

*Let $G$ be a finite nilpotent group. Then $G$ is filled if and only if $G$ is either an elementary abelian 2-group or one of $C_3$, $C_5$, $D_8$, $D_8 \times C_2$, $D_8 * Q_8$ or $(D_8 * Q_8) \times C_2$.*

# Conjecture on finite filled groups

**Conjecture (A, Erskine, Hart '18)**

Any finite filled group is either an elementary abelian 2-group or one of $C_3$, $C_5$, $D_6$, $D_8$, $D_{10}$, $D_{12}$, $D_{14}$, $D_8 \times C_2$, $D_{22}$, $D_8 * Q_8$ and $(D_8 * Q_8) \times C_2$.

We have searched for filled groups in all groups of order up to 2000, and could only find those given in the conjecture.

# On filled soluble groups

**Theorem (A' 2018)**

*Any finite filled group of order $p^n q^m$ for primes $p$ and $q$, and $m, n \in \mathbb{N}$ is either an elementary abelian 2-group or one of $D_6$, $D_8$, $D_{10}$, $D_{12}$, $D_{14}$, $D_8 \times C_2$, $D_{22}$, $D_8 * Q_8$ or $(D_8 * Q_8) \times C_2$.*

**Theorem (A' 2018)**

*No group of order $2pq$ (for primes $p, q$ with $2 < p < q$) is filled.*

**Theorem (A' 2020)**

*$C_2^3$, $D_8$ and $D_{12}$ are the only filled groups of order $pqr$ for primes $p$, $q$ and $r$.*

**Work in progress:** The only finite filled soluble groups are the ones mentioned in the conjecture.

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