Steiner triple systems: a model theoretic viewpoint

Silvia Barbina[1]

joint work with Enrique Casanovas[2]

[1]Università di Camerino

[2]Universitat de Barcelona

OU Discrete Mathematics Seminar

12 April 2023

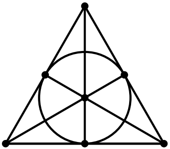
Steiner triple systems

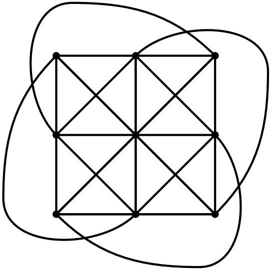
Definition

A **Steiner triple system** (STS) of order n is a pair $(V, \mathcal{B})$ where:

- V is a set of n elements;
- $\mathcal{B}$ is a collection of 3-element subsets of V (the **blocks**) such that any two $x, y \in V$ are contained in exactly one block.

A set V with a collection of 3-element subsets is a **partial STS** if any two elements of V belong to at most one block.





Finite STSs appear in

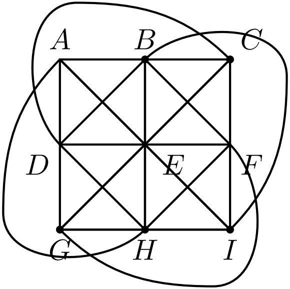
- combinatorial design theory (they are balanced incomplete block designs)
- coding theory
- group theory (e.g.: every finite group is the automorphism group of a finite Steiner triple system)
- ...

An application

A pharmaceutical research team needs to test interactions among 9 drugs.

Ideally: drugs are compared in sets of 3.

In practice: select groups of three so that each pair of drugs appears in exactly one group.



A	B	C	A	...	F
D	E	F	B	...	H
G	H	I	C	...	A

Cardinality of STSs

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod{6}$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

Extension properties

- Every finite partial STS can be embedded in a finite STS.
- Every infinite partial STS can be embedded in an STS of the same cardinality.

First-order structures

Each of

- the ring of integers, $\mathbb{Z}$
- the ordered integers
- the field of rational numbers, $\mathbb{Q}$
- the ordered rationals

can be viewed as a *structure*, i.e. a set plus

- a collection of functions (unary, binary, ...)
- a collection of relations
- a collection of distinguished elements (constants)

(all of these possibly empty)

Languages

Each of these structures is described by choosing an appropriate **language**, i.e. a set of

- function symbols (various arities),
- relation symbols (various arities),
- constant symbols

$L_{\mathrm{rng}} = \{+, \cdot, -, 0, 1\}$ is the language of rings.
$L_{\mathrm{ord}} = \{<\}$ is the language of ordered sets.

An L-**structure** is a structure where each of the symbols in L is interpreted by a suitable function, relation or distinguished element.

Notation: $(\mathbb{Z}, +, \cdot, -, 0, 1)$, $(\mathbb{Z}, <)$, $(\mathbb{Q}, +, \cdot, -, 0, 1)$, etc.

Choice of language for STS

A Steiner triple system can be regarded as a structure in two ways:

- a set A with a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block

or

- a set A with a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

We choose the latter.

If blocks are viewed as the graph of a binary operation, STS are often called **Steiner quasigroups**.

The language L_{Sq} contains a single binary operation.

A **subsystem** of an STS is a subset which is closed under the operation (so an STS in its own right).

An STS A is said to be **generated by** $\{a_i : i \in I\} \subseteq A$ if A is the smallest STS containing the a_i. Then we write

$$A = \langle a_i : i \in I \rangle .$$

Terms and formulas

If L is a language, L-**terms** are built using

- variables
- constants of L
- function symbols of L.

L-**formulas** are built using

- the symbols of L
- parentheses (and)
- the equality symbol $=$
- variables $x_0, x_1, x_2, \ldots, x_n, \ldots$
- logical connectives $\wedge, \vee, \neg, \rightarrow$
- quantifiers $\forall, \exists$.

Examples:

- $x \cdot (y \cdot x)$
- $((x \cdot y) \cdot z) \cdot (z \cdot v)$

are L_{Sq}-terms, and

- $x \cdot (y \cdot z) = x \cdot y$
- $\forall xyz \, [x \cdot (y \cdot z) = x \cdot y \to y = z]$

are L_{Sq}-formulas.

Two important restrictions:

1. formulas and terms are *finite* strings of symbols
2. quantifiers only range over elements of a structure, not over subsets.

Theories

A **sentence** in a first-order language L is a formula with no free variables.

For an L-structure M, the set $\mathrm{Th}(M)$ is the set of sentences that hold in M. This is called the **first-order theory** of M.

More generally, a **theory** is a set of L-sentences. For example, T_{Sq} is the theory that consists of

1. $\forall xy\,(x \cdot y = y \cdot x)$
2. $\forall x\,(x \cdot x = x)$
3. $\forall xy\,(x \cdot (x \cdot y) = y)$.

If M is an STS, then $T_{\mathrm{Sq}} \subseteq \mathrm{Th}(M)$.

Expressiveness of the first-order theory

If M is an STS and $M = \langle a_i : i \in I \rangle$, then every element of M can be written as $t(a_0, \ldots, a_n)$ for some term $t(x_0, \ldots, x_n)$.

Some properties of (infinite) STSs can be expressed by the first-order theory, for instance

- $\forall xyz \, (x \cdot y = x \cdot z \to y = z)$

Others cannot. For instance:

- being finitely generated: naively expressed by an infinite disjunction, which is not allowed, or
- every subsystem is finitely generated: this would naively be second order, i.e. involve quantification over subsystems.

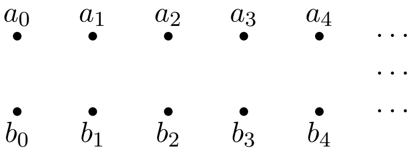
Why first-order theories?

▸ Universe

Some regions are defined via the presence or absence of certain combinatorial configurations in models.

A theory has the **order property** if and only if there is a formula $\phi(x, y)$ and two infinite sequences $\langle a_i \in \omega \rangle$ and $\langle b_i \in \omega \rangle$ such that

$$\phi(a_i, b_j) \iff i \leq j\,.$$



A theory is **stable** if and only if it does *not* have the order property.

Infinite Steiner triple systems

- Projective and affine countable STS (ω-categorical)
  - projective: defined on the projective space of dimension ω over $\mathbb{F}_2$
  - affine: defined on the affine space of dimension ω over $\mathbb{F}_3$

  via $B(x, y, z)$ iff $x + y + z = 0$
- the countable homogeneous universal STS
- **Chicot, Grannel, Griggs, Webb** 2010: 2^ω sparse countable STS
- **Horsley Webb** 2021: 2^ω countable ultrahomogeneous triple systems omitting specified subsystems
- **Baldwin** 2022: 2^ω strongly minimal STS

A Fraïssé limit

The class $\mathcal{C}$ of all finite Steiner triple systems

- has the *Joint Embedding* and *Amalgamation Properties*
- is closed under substructures
- has countably many elements up to isomorphism.

By Fraïssé's Theorem, there is a unique (up to isomorphism) countable Steiner triple system M_F that is

- universal for finite STSs (every finite STS embeds in M_F)
- ultrahomogeneous (every isomorphism between finite subsystems of M_F extends to an automorphism of M_F).

M_F is the **Fraïssé limit** of $\mathcal{C}$.

M_F is locally finite: each finite subset of M_F generates a finite substructure.

Axiomatising Th(M_F)

Definition

Let B be a finite partial STS. Then

- δ_B is a formula that fully describes the elements and blocks of B
- $A \subseteq B$ is **relatively closed** in B if for every $a, b \in A$ and $c \in B$, if $a \cdot b = c$ then $c \in A$.

Axiomatising $\mathrm{Th}(M_F)$

Definition

If B is a finite partial STS and $A \subseteq B$ a relatively closed subset, then

$$\phi_{(A,B)} = \forall \bar{x} \left(\delta_A(\bar{x}) \rightarrow \exists \bar{y}\, \delta_B(\bar{x}, \bar{y}) \right).$$

Let $\Delta = \{\phi_{(A,B)} : B$ is a finite partial STS and $A \subseteq B$ is a relatively closed subset$\}$.

Let $T_{M_F} = \Delta \cup T_{\mathrm{Sq}}$ (where T_{Sq} includes

- $\forall xy \left(x \cdot y = y \cdot x \right)$
- $\forall x \left(x \cdot x = x \right)$
- $\forall xy \left(x \cdot (x \cdot y) = y \right)$.)

Properties (SB, Casanovas)

The theory T_{M_F}

- axiomatizes the theory of M_F
- has *quantifier elimination*
- excludes finitely generated infinite subsystems
- is TP_2
- is NSOP_1
- ...

TP_2

Definition

A formula $\varphi(\overline{x};\overline{y})$ has the *tree property of the second kind* (TP_2) in T if in a large, saturated model of T there is an array of tuples $(\overline{a}_{ij} \mid i,j<\omega)$ and some natural number k such that

- for each $i<\omega$ the set $\{\varphi(\overline{x},\overline{a}_{ij}) \mid j<\omega\}$ is k-inconsistent
- for each $f:\omega\to\omega$ the path $\{\varphi(\overline{x},\overline{a}_{if(i)}) \mid i<\omega\}$ is consistent.

We say that T is TP_2 if some formula has TP_2 in T.

$$\begin{array}{ccccc} \overline{a}_{00} & \overline{a}_{01} & \overline{a}_{02} & \overline{a}_{03} & \cdots \\ \overline{a}_{10} & \overline{a}_{11} & \overline{a}_{12} & \overline{a}_{13} & \cdots \\ \overline{a}_{20} & \overline{a}_{21} & \overline{a}_{22} & \overline{a}_{23} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

Proposition

The formula

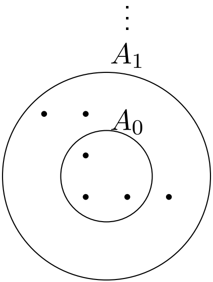
$$\varphi(x; y_1, y_2, y_3) \equiv \quad x = (y_1 \cdot (y_2 \cdot (y_3 \cdot x)))$$

has TP_2 in T_{M_F}.

A free construction

A construction of this kind is used in the proof of some of the properties of $\mathrm{Th} M_F$.

It is, in fact, one of the STSs built in Chicot, Grannel, Griggs, Webb 2010.



We can start from $A = \{a_i : i \in I\}$ where I is countable and no three points of A form a block.

Characterizations of freeness

See Siebenmann (1965) and Paolini & Hyttinen (2020) on free projective planes.

Definition

A **free STS on λ generators** is an STS for which there is a chain $\{A_i : i < \omega\}$ of subsets such that

(i) $A_0 = \{a_i : i < \lambda\}$

(ii) if $i \neq j$, then the product $a_i a_j$ is not defined in A_0

(iii) $A_i \subseteq A_{i+1}$

(iv) if $a \in A_{i+1} \setminus A_i$ then a lies in exactly one block of A_{i+1}.

Fact

A subsystem of a free STS is free.

Definition

A partial STS P is **unconfined**, or **contains no confined configuration**, if for all finite $P' \subseteq P$ there is $p \in P'$ that belongs to at most one block in P', that is, p is the product of at most two elements of P'.

We can express unconfinedness via a set of first-order sentences.

Fact

If S is a free STS on λ generators, then S is unconfined.

Proof

Suppose $S = \bigcup_{i \in \omega} A_i$ and $P \subseteq S$ is finite. Then $P \subseteq A_i$ for some i.

Question

A finitely generated STS which is unconfined is free. Is every unconfined STS free?

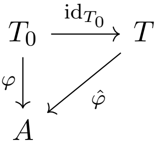
Some terminology and techniques from universal algebra.

Definition

Let $\mathcal{C}$ be a class of STS. An STS T generated by $T_0 = \{t_i : i < \alpha\}$ has the **universal mapping property** (UMP) **for** $\mathcal{C}$ if for every $A \in \mathcal{C}$ and every map

$$\varphi : T_0 \to A$$

there is a homomorphism $\hat{\varphi} : T \to A$ that extends φ.



Cf. the UMP for free groups.

Fact

(i) A free STS with λ generators has the UMP for the class of all STS.

(ii) If T is an STS on λ generators with the UMP for the class of all STS, then T is the free STS on λ generators.

The generators are said to be a **free base** for the STS.

For *finitely* generated STS F, tfae

- F is free
- F has UMP for the class of all Steiner quasigroups
- F is unconfined.

Fact

If M and N are free STS with free bases A, B respectively, then

$$M \cong N \iff |A| = |B|.$$

What about $\mathrm{Th}(M)$ and $\mathrm{Th}(N)$?

Conjecture

$\mathrm{Th}(M)$ is independent of $|A|$, that is, if M is the free STS on λ generators and N is the free STS on μ generators, then $\mathrm{Th}(M) = \mathrm{Th}(N)$.

A notable example: free groups!