Limits of permutations
(and some other discrete objects)

David Bevan
University of Strathclyde

Discrete Mathematics Seminar
The Open University

16th December 2020

Permutations

- Permutation of length n: an ordering on $1, \ldots, n$.

Example

$$\pi = 314592687$$

Permutations

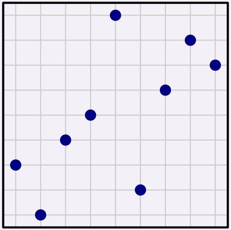
- Permutation of length n: an ordering on $1, \ldots, n$.

Example

$$\pi = 314592687$$

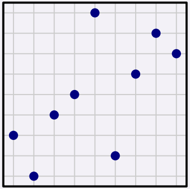
- Graphical perspective: plot the points $(i, \pi(i))$.
- One point in each row; one point in each column.

Example (314592687)

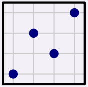


Containment and Pattern Density

Example



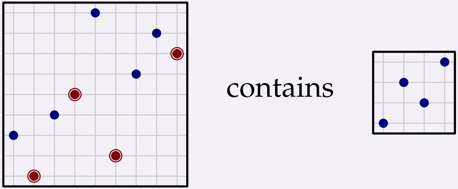
contains



$\pi =$ 314592687 contains $\tau =$ 1324

Containment and Pattern Density

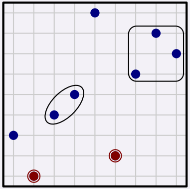
Example



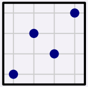
$\pi =$ 314592687 contains $\tau =$ 1324

Containment and Pattern Density

Example



contains

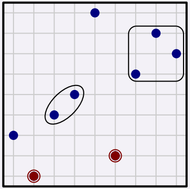


$\pi =$ 314592687 contains $\tau =$ 1324

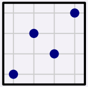
- Six occurrences of 1324 in 314592687: $\nu(\tau, \pi) = 6$

Containment and Pattern Density

Example



contains



$\pi = 314592687$ contains $\tau = 1324$

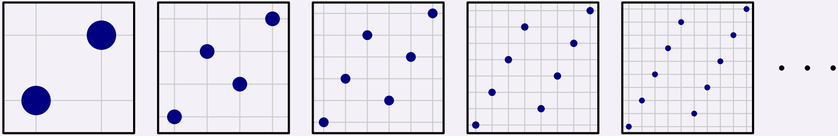
- Six occurrences of 1324 in 314592687: $\nu(\tau, \pi) = 6$
- $\binom{9}{4} = 126$ choices of four points from π
- Density of 1324 in 314592687 is $\frac{6}{126} = \frac{1}{21}$: $\rho(\tau, \pi) = \frac{1}{21}$

Definition (pattern density)

$\rho(\tau, \pi) = \nu(\tau, \pi)/\binom{n}{k}$, where $n = |\pi|$ and $k = |\tau|$

Limits?

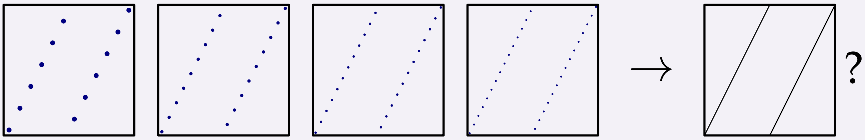
Example



$$\pi_j = 13579\ldots(2j-1)2468\ldots(2j)$$

Limits?

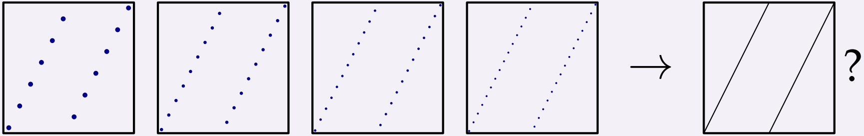
Example



$$\pi_j = 13579\ldots(2j-1)2468\ldots(2j)$$

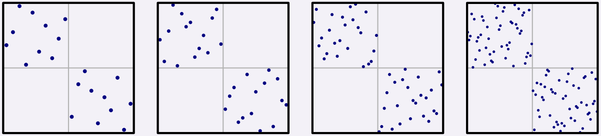
Limits?

Example



$$\pi_j = 13579\ldots(2j-1)2468\ldots(2j)$$

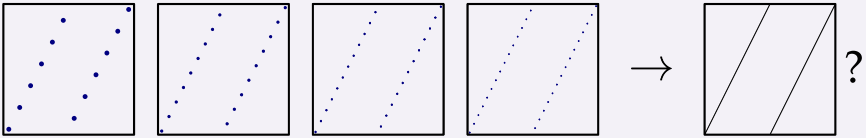
Example (random)



$$\varphi_j = \boldsymbol{\sigma}_j \ominus \boldsymbol{\sigma}_j$$

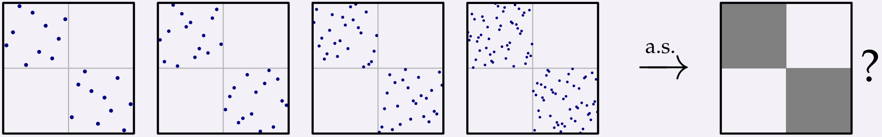
Limits?

Example



$$\pi_j = 13579\ldots(2j-1)2468\ldots(2j)$$

Example (random)



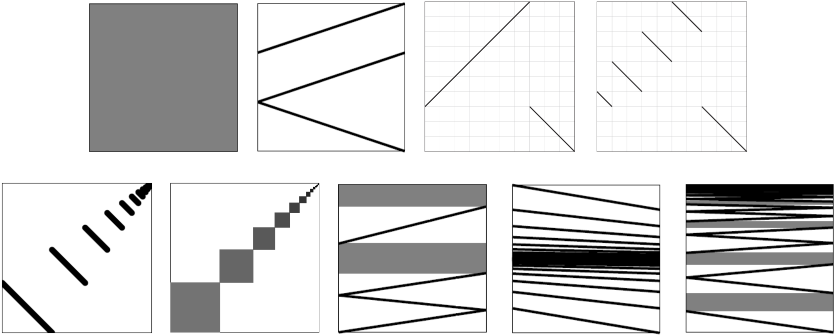
$$\varphi_j = \boldsymbol{\sigma}_j \ominus \boldsymbol{\sigma}_j$$

Permutons

Definition (permuton)

Probability measure μ on the σ-algebra of Borel sets of the unit square $[0,1]^2$ such that μ has uniform marginals:

$$\mu([a,b] \times [0,1]) = \mu([0,1] \times [a,b]) = b - a \text{ for every } 0 \leqslant a \leqslant b \leqslant 1$$

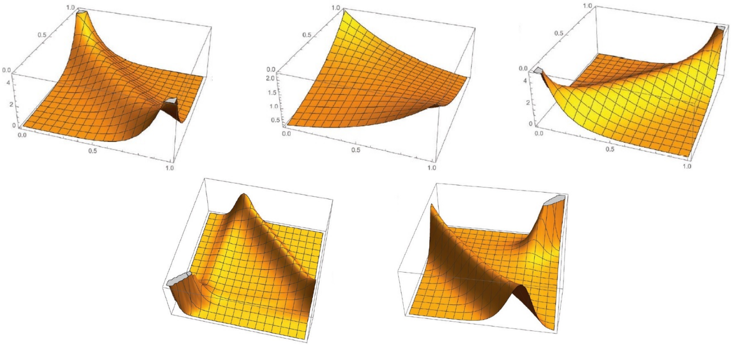


Permutons

Definition (permuton)

Probability measure μ on the σ-algebra of Borel sets of the unit square $[0,1]^2$ such that μ has uniform marginals:

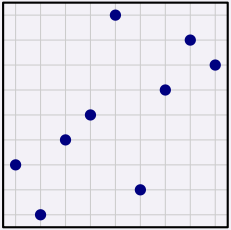
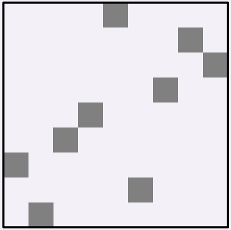
$$\mu([a,b]\times[0,1]) \;=\; \mu([0,1]\times[a,b]) \;=\; b-a \;\text{ for every }\; 0\leqslant a\leqslant b\leqslant 1$$



Permutation permuton

- Permuton μ_π corresponding to permutation π

Example (314592687)

$\pi =$  $\mu_\pi =$ 

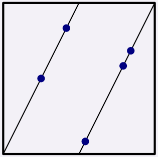
- Mass for each point: $1/n$
- Small square area: $1/n^2$
- Density ("height"): n

Sampling from a permuton

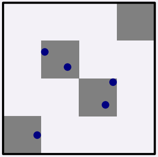
A permuton μ is a probability distribution over the unit square

- Can use to randomly sample k points
- Probability of sharing a coordinate is zero (uniform marginals)
- Order of y-coordinates gives a μ-random permutation of length k

Two examples



25134



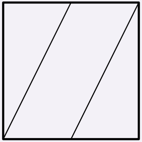
15423

Permuton pattern density

Definition (permuton pattern density)

$\rho(\tau, \mu) = \mathbb{P}\big[\text{a } \mu\text{-random permutation of length } k \text{ equals } \tau\big] \quad (k = |\tau|)$

Example ($\mu_{//}$)

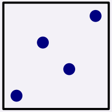


$$\rho(12, \mu_{//}) = \frac{3}{4}$$

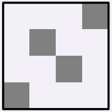
Pattern density

- In general, $\rho(\tau, \pi) \neq \rho(\tau, \mu_\pi)$

Example (1324)



$$\rho(12, 1324) = \frac{5}{6}$$



$$\rho(12, \mu_{1324}) = \frac{3}{4}$$

- But they don't differ too much:

Lemma

$$\left|\rho(\tau, \pi) - \rho(\tau, \mu_\pi)\right| \leqslant \frac{1}{n}\binom{k}{2}, \text{ where } n = |\pi| \text{ and } k = |\tau|.$$

Global convergence

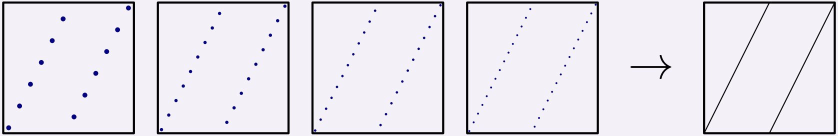
Definition (global convergence)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is (globally) convergent if $\rho(\tau, \pi_j)$ converges for every permutation τ.

The permuton μ is the limit if $\lim_{j\to\infty} \rho(\tau, \pi_j) = \rho(\tau, \mu)$ for every τ.

- General principle: asymptotic proportions of substructures

Example



$\rho(12, \pi_j) = \frac{3j-1}{4j-2} \to \frac{3}{4}$

$\rho(12, \mu_{//}) = \frac{3}{4}$

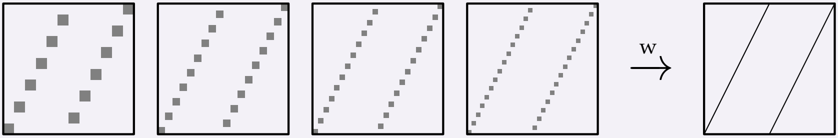
Every convergent sequence has a permuton limit

Theorem (Hoppen, Kohayakawa, Moreira, Ráth, Sampaio)

Convergence of pattern densities is equivalent to *weak convergence*† of permutons: $\forall \tau \in \mathcal{S},\ \rho(\tau, \mu_j) \to \rho(\tau, \mu) \iff \mu_j \xrightarrow{\mathrm{w}} \mu.$

Since $\lim_{n\to\infty} \left|\rho(\tau, \pi_j) - \rho(\tau, \mu_{\pi_j})\right| = 0$, the sequence (π_j) converges to the weak limit of the sequence (μ_{π_j}).

Example



$\mu_{\pi_j} \xrightarrow{\mathrm{w}} \mu_{//}$

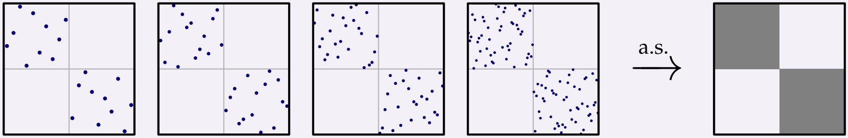
† $\mathbb{P}_j[A] \to \mathbb{P}[A]$ if $\mathbb{P}[\partial A] = 0$

Every permuton is the limit of a sequence

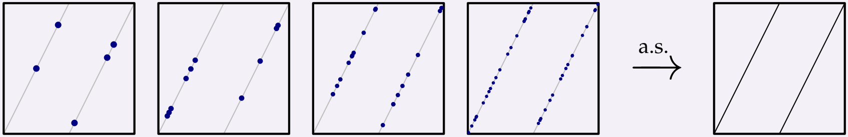
Theorem (Hoppen, Kohayakawa, Moreira, Ráth, Sampaio)

If π_j is a μ-random permutation of length j, then $\pi_j \xrightarrow{\text{a.s.}} \mu$.

Example



Example



Interlude: Graph limits

Definition (density)

If G and H are graphs, $\rho(H, G)$ is the probability that $|H|$ vertices of G induce a subgraph isomorphic to H.

Definition (global convergence)

If $|G_j| \to \infty$, then $(G_j)_{j \in \mathbb{N}}$ is (globally) convergent if $\rho(H, G_j)$ converges for every graph H.

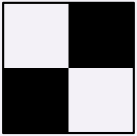
Definition (graphon)

Measurable function $W : [0, 1]^2 \to [0, 1]$ symmetric about $y = x$.

- continuous analogue of adjacency matrix
- W-random graph of order k: choose k points $x_1, \ldots, x_k$ uniformly at random and join vertices i and j with probability $W(x_i, x_j)$

Interlude: Graph limits

Example (complete balanced bipartite graphs)

$$K_{j,j} \longrightarrow$$ 

- If φ is a measure-preserving map $[0,1] \to [0,1]$, then $W^\varphi(x,y) = W(\varphi(x),\varphi(y))$ is "the same" graphon as W.
- continuous analogue of permuting matrix rows/columns

Theorem (Lovasz, Szegedy)

If $(G_j)_{j\in\mathbb{N}}$ is convergent, then there is a graphon W such that $G_j \to W$.

Theorem

If G_j is a W-random graph of order j, then $G_j \xrightarrow{\text{a.s.}} W$.

Interlude: Latin square limits

- Pattern: $k \times \ell$ matrix containing each of $1, \ldots, k\ell$
- Occurrence: submatrix order-isomorphic to pattern

Example (Fisher stained glass window, Gonville & Caius, Cambridge)

contains $\begin{matrix} 1 & 3 \\ 2 & 6 \\ 5 & 4 \end{matrix}$

- Density $\rho(A, L)$: Probability that $k \times \ell$ submatrix of L matches A
- Convergence: $|L_j| \to \infty$ and $\rho(A, L_j)$ converges for all patterns A

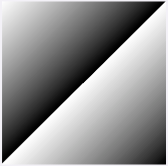
Interlude: Latin square limits

Definition (Latinon)

A pair (W, f) consisting of a measurable function $W : [0,1]^2 \to \mathcal{B}[0,1]$ (Borel probability measures on $[0,1]$) satisfying a symmetric uniformity condition, and a measure-preserving function $f : [0,1] \to [0,1]$.

Example (convergence of cyclic Latin squares to cyclic Latinon)

$$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{matrix} \quad \longrightarrow$$

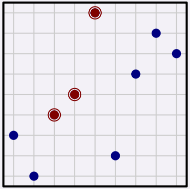
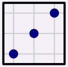


Theorems (Garbe, Hancock, Hladký, Sharifzadeh)

Every convergent sequence of Latin squares has a Latinon limit.
Every Latinon is a limit of a convergent sequence of Latin squares.

Consecutive patterns

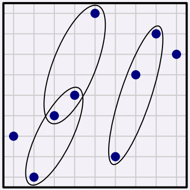
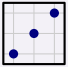
Example

 contains  consecutively

$\pi =$ 314592687 contains $\tau =$ 123 consecutively

Consecutive patterns

Example

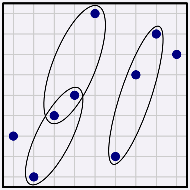
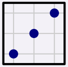
 contains  consecutively

$\pi =$ 314592687 contains $\tau =$ 123 consecutively

- Three consecutive occurrences of 123 in 314592687: $\nu_{\mathsf{c}}(\tau, \pi) = 3$

Consecutive patterns

Example

 contains  consecutively

$\pi =$ 314592687 contains $\tau =$ 123 consecutively

- Three consecutive occurrences of 123 in 314592687: $\nu_{\mathsf{c}}(\tau, \pi) = 3$
- $9 + 1 - 3 = 7$ choices of three consecutive points from π
- Density of consecutive 123 in 314592687 is $\frac{3}{7}$: $\rho_{\mathsf{c}}(\tau, \pi) = \frac{3}{7}$

Definition (consecutive pattern density)

$\rho_{\mathsf{c}}(\tau, \pi) = \nu_{\mathsf{c}}(\tau, \pi)/(n + 1 - k)$, where $n = |\pi|$ and $k = |\tau|$

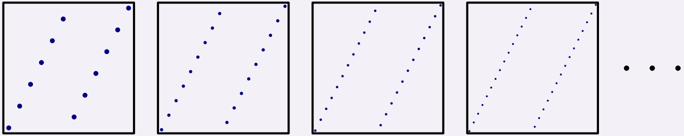
Local convergence

Definition (local convergence)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is locally convergent if $\rho_{\mathsf{c}}(\tau, \pi_j)$ converges for every permutation τ.

- Asymptotic proportions of local substructures

Example



$$\rho_{\mathsf{c}}(12, \pi_j) = \frac{2j-2}{2j-1} \to 1$$

Local convergence

- Local limit of locally convergent sequence of permutations: shift-invariant random infinite rooted permutation (SIRIRP)

Theorem (Borga)

Every locally convergent sequence of permutations has a SIRIRP as a local limit.

Theorem (Borga)

Every SIRIRP is the local limit of some locally convergent sequence of permutations.

Local convergence

- Local limit of locally convergent sequence of permutations: shift-invariant random infinite rooted permutation (SIRIRP)

Theorem (Borga)

Every locally convergent sequence of permutations has a SIRIRP as a local limit.

Theorem (Borga)

Every SIRIRP is the local limit of some locally convergent sequence of permutations.

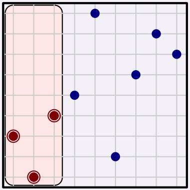
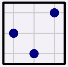
Brief aside: local convergence for graphs

- Benjamini-Schramm convergence; analytic limits: graphings

Patterns at a given scale

- Looking through a window

Example (window of width 3)

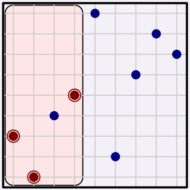
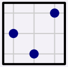
π =  contains 15 occurrences of τ = 

- 1 has width 3 (consecutive)

Patterns at a given scale

- Looking through a window

Example (window of width 4)

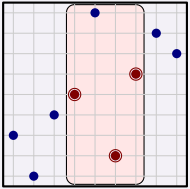
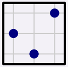
π =  contains 15 occurrences of τ = 

- 1 has width 3 (consecutive); 2 have width 4

Patterns at a given scale

- Looking through a window

Example (window of width 4)

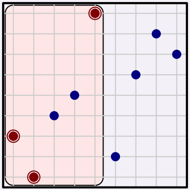
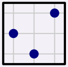
$\pi =$  contains 15 occurrences of $\tau =$ 

- 1 has width 3 (consecutive); 2 have width 4

Patterns at a given scale

- Looking through a window

Example (window of width 5)

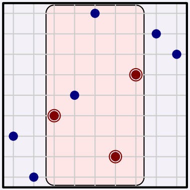
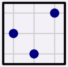
$\pi =$  contains 15 occurrences of $\tau =$ 

- 1 has width 3 (consecutive); 2 have width 4; 3 have width 5

Patterns at a given scale

- Looking through a window

Example (window of width 5)

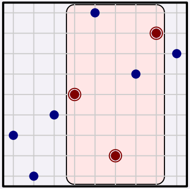
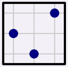
$\pi =$  contains 15 occurrences of $\tau =$ 

- 1 has width 3 (consecutive); 2 have width 4; 3 have width 5

Patterns at a given scale

- Looking through a window

Example (window of width 5)

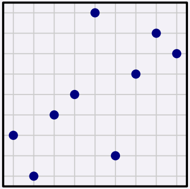
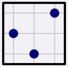
π =  contains 15 occurrences of τ = 

- 1 has width 3 (consecutive); 2 have width 4; 3 have width 5

Patterns at a given scale

- Looking through a window

Example (window of width 5)

$\pi =$  contains 15 occurrences of $\tau =$ 

- 1 has width 3 (consecutive); 2 have width 4; 3 have width 5
- 6 have width at most 5: $\nu_5(\tau, \pi) = 6$
- 34 choices of three points with width at most 5
- Density of τ in π at scale 5 is $\frac{6}{34} = \frac{3}{17}$: $\rho_5(\tau, \pi) = \frac{3}{17}$

Patterns at a given scale

Definition (pattern density at scale f)

$\rho_f(\tau,\pi) = \nu_f(\tau,\pi)/\binom{n}{k}_f$, where $n = |\pi|$ and $k = |\tau|$ and

$$\binom{n}{k}_f = \sum_{w=k}^{f}(n+1-w)\binom{w-2}{k-2}$$

- Typically, the scale (width of window) $f = f(n)$ depends on n

Examples

- Global: $\rho_n(\tau,\pi) = \rho(\tau,\pi)$, where $n = |\pi|$
- Local: $\rho_k(\tau,\pi) = \rho_{\mathsf{C}}(\tau,\pi)$, where $k = |\tau|$

Definition (scaling function)

$f : \mathbb{N} \to \mathbb{N}$ is a scaling function if $1 \ll f(n) \leqslant n$

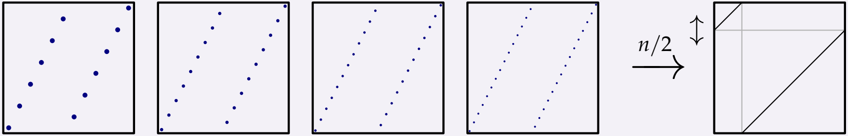
Convergence at a given scale

Definition (convergence at scale f)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is convergent at scale f if $\rho_f(\tau, \pi_j)$ converges for every permutation τ.

- Asymptotic proportions of substructures visible at scale f
- Limit object: random permuton (i.e. distribution over permutons)
- Question: Which distributions?

Example ($f = n/2$)



$$\rho_{n/2}(12, \pi_j) = \frac{2}{3} \quad (n = |\pi_j| = 2j)$$

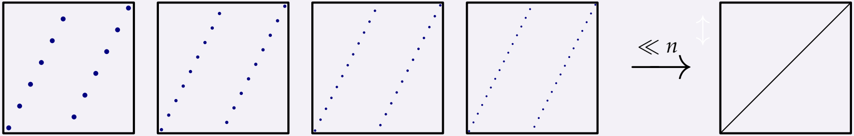
Convergence at a given scale

Definition (convergence at scale f)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is convergent at scale f if $\rho_f(\tau, \pi_j)$ converges for every permutation τ.

- Asymptotic proportions of substructures visible at scale f
- Limit object: random permuton (i.e. distribution over permutons)
- Question: Which distributions?

Example ($f \ll n$)



$$\rho_f(12, \pi_j) \to 1, \text{ if } f \ll n$$

Scalable convergence

Definition (scalable convergence)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is scalably convergent if, for every $\tau \in \mathcal{S}$ there exists ρ_τ such that $\rho_f(\tau, \pi_j) \to \rho_\tau$ for every $f \ll n$.

- Limit doesn't depend on scaling function ($\ll n$)

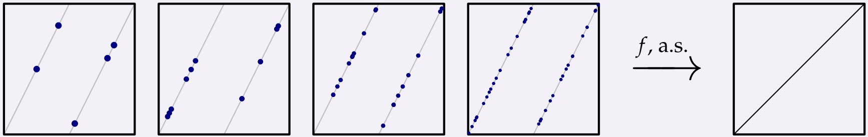
Scalable convergence

Definition (scalable convergence)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is scalably convergent if, for every $\tau \in \mathcal{S}$ there exists ρ_τ such that $\rho_f(\tau, \pi_j) \to \rho_\tau$ for every $f \ll n$.

- Limit doesn't depend on scaling function ($\ll n$)

Example (μ-random sequence)



- Scalable limit $\neq$ global limit

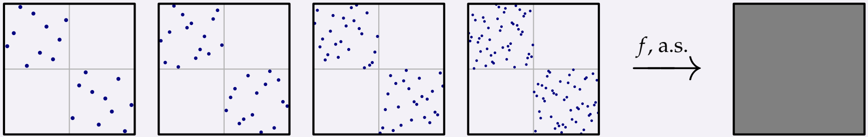
Scalable convergence

Definition (scalable convergence)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is scalably convergent if, for every $\tau \in \mathcal{S}$ there exists ρ_τ such that $\rho_f(\tau, \pi_j) \to \rho_\tau$ for every $f \ll n$.

- Limit doesn't depend on scaling function ($\ll n$)

Example (μ-random sequence)



- Scalable limit $\neq$ global limit

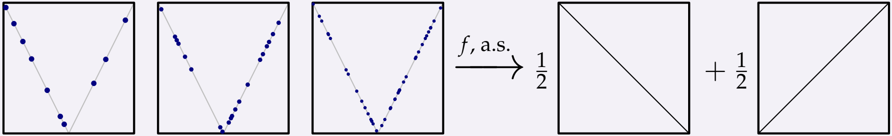
Scalable convergence

Definition (scalable convergence)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is scalably convergent if, for every $\tau \in \mathcal{S}$ there exists ρ_τ such that $\rho_f(\tau, \pi_j) \to \rho_\tau$ for every $f \ll n$.

- Limit doesn't depend on scaling function ($\ll n$)

Example (μ-random sequence)



- Scalable limit $\neq$ global limit

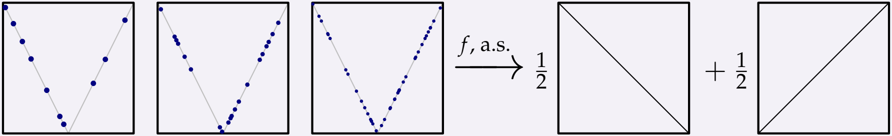
Scalable convergence

Definition (scalable convergence)

If $|\pi_j| \to \infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is scalably convergent if, for every $\tau \in \mathcal{S}$ there exists ρ_τ such that $\rho_f(\tau, \pi_j) \to \rho_\tau$ for every $f \ll n$.

- Limit doesn't depend on scaling function ($\ll n$)

Example (μ-random sequence)



- Scalable limit $\neq$ global limit

Conjecture (B.)

For any permuton μ, a μ-random sequence is scalably convergent a.s.

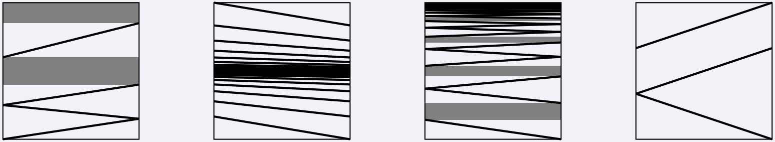
Tiered permutons

- For which permutons μ is a μ-random sequence scalably convergent to μ itself?

Tiered permutons

- For which permutons μ is a μ-random sequence scalably convergent to μ itself?
- Tiered permuton: countable uniform or diagonal tiers

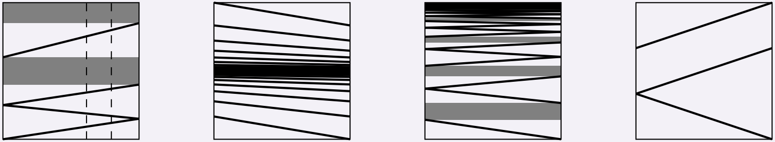
Some tiered permutons



Tiered permutons

- For which permutons μ is a μ-random sequence scalably convergent to μ itself?
- Tiered permuton: countable uniform or diagonal tiers

Some tiered permutons



- If μ is tiered, any (re-scaled) vertical slice $\mu_{[a,b]} = \mu$.
- If μ is tiered, a μ-random sequence is scalably convergent to μ.

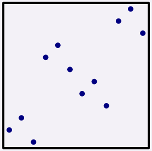
Conjecture (B.)

If a μ-random sequence is scalably convergent to μ then μ is tiered.

Independence of limits at different scales

- Substitution $\sigma[\tau]$: replace each point of σ with a copy of τ

Example

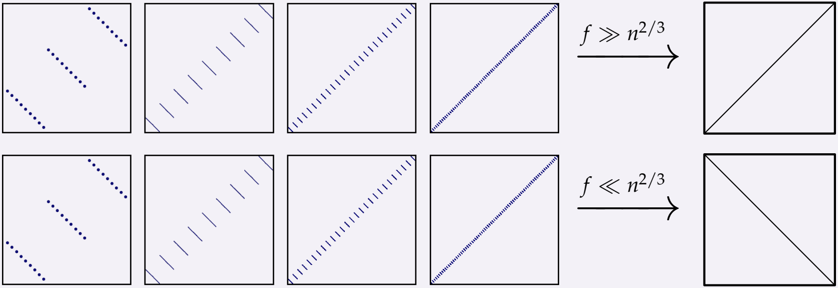
$1324[231] =$ 

Independence of limits at different scales

- Substitution $\sigma[\tau]$: replace each point of σ with a copy of τ

Example ($\pi_j = \iota_j[\delta_{j^2}]$; $n = j^3$)

$\iota_k = 123\ldots k \qquad \delta_k = k\ldots 321$

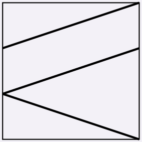


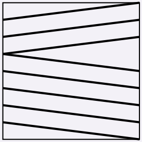
$\pi_j \xrightarrow{\sqrt{n}}$  $\qquad \pi_j \xrightarrow{n^{3/4}}$ 

Independence of limits at infinitely many scales

Some tiered permutons

$\mu_{p,q}$: q equal tiers; upper p increasing; lower $q-p$ decreasing

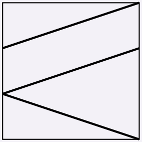
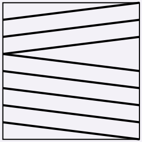
$\mu_{2,3} =$ 

$\mu_{3,8} =$ 

Independence of limits at infinitely many scales

Some tiered permutons

$\mu_{p,q}$: q equal tiers; upper p increasing; lower $q-p$ decreasing

$\mu_{2,3} =$  $\mu_{3,8} =$ 

Infinitely many limits

- We can construct a sequence of permutations $(\zeta_j)_{j\in\mathbb{N}}$ such that, for each irreducible $p/q \in \mathbb{Q}\cap(0,1]$, we have

$$\zeta_j \xrightarrow{n^{p/q}} \mu_{p,q}.$$

B., Independence of permutation limits at infinitely many scales. arXiv:2005:11568

Thanks! And Merry Christmas!