Labelled well-quasi-order for permutation classes

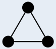
Robert Brignall

Joint work with Vince Vatter (U. Florida)

Combinatorics Seminar, OU, 2 December 2020

Embedding graphs

Question (easy)

Can I delete vertices (and incident edges) from  *to produce*  *?*

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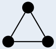
Can I delete vertices (and incident edges) from  *to produce*  *?*

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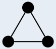
Can I delete vertices (and incident edges) from  *to produce*  *?*

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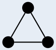
Can I delete vertices (and incident edges) from  *to produce*  *?*

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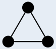
Can I delete vertices (and incident edges) from  *to produce*  *?*

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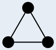
Can I delete vertices (and incident edges) from  *to produce*  *?*

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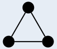
Can I delete vertices (and incident edges) from  *to produce*  *?*

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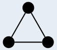
Can I delete vertices (and incident edges) from  *to produce*  *?*

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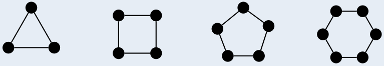
Is any graph in the following (infinite) list an induced subgraph of another?



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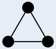
Can I delete vertices (and incident edges) from  *to produce*  *?*

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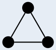
Can I delete vertices (and incident edges) from  *to produce*  *?*

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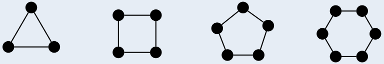 . . .

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Labelled embeddings

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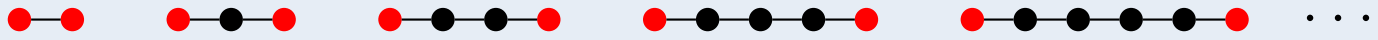

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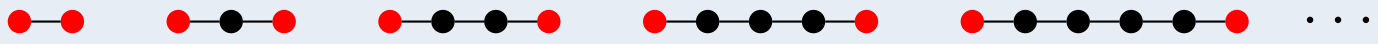

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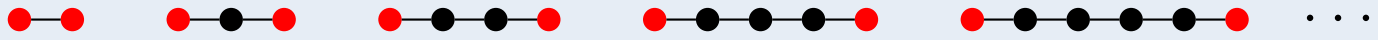

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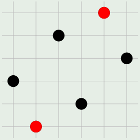
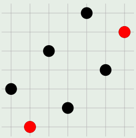
A similar phenomenon in permutations



3 1 5 2 6 4



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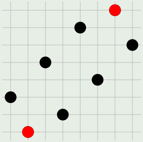

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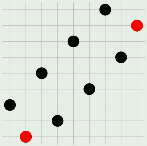

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§1 Combinatorial structures

Relational structures

A relational structure comprises

- A ground set
- One or more relations

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- Ground set: vertices V
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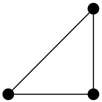

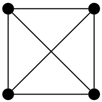

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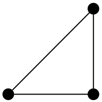
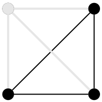
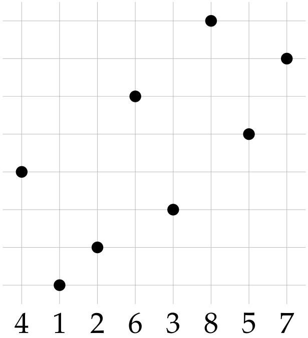
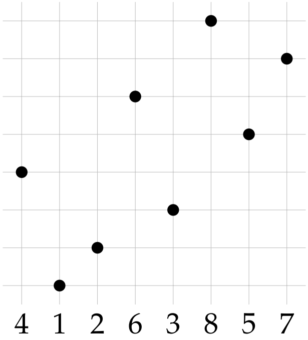

$\leq_{\text{ind}}$



4 1 2 6 3 8 5 7

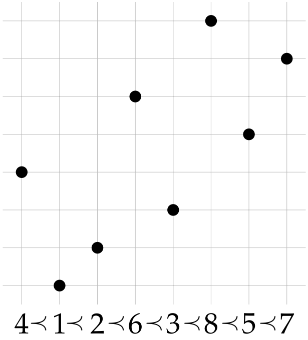


Permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$

- Ground set: entries $\{1, 2, \ldots, n\}$ (or any set of size n)
- Relations: two linear orders, $<$ and $\prec$:

$$1 < 2 < \cdots < n$$
$$\pi(1) \prec \pi(2) \prec \cdots \prec \pi(n)$$

($\prec$ is the 'reading order' of the permutation)



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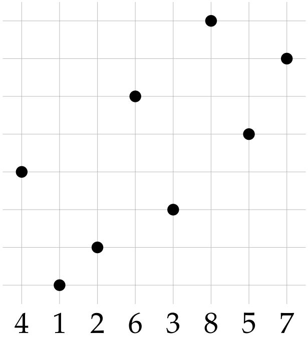
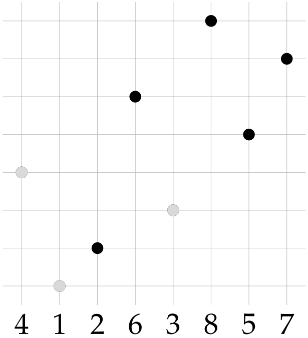


- Induced subpermutation ordering: containment



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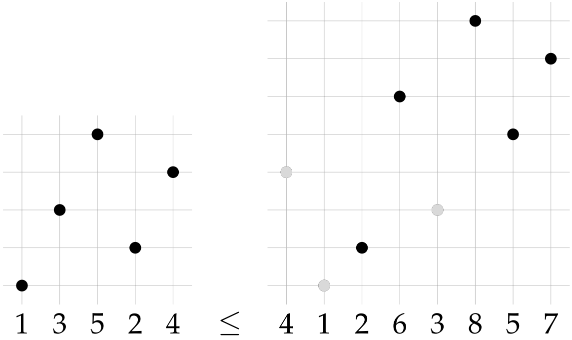

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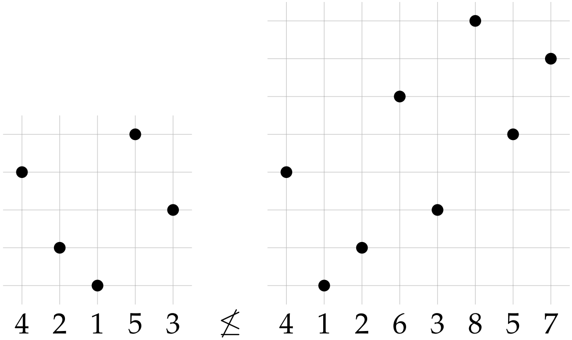

- Induced subpermutation ordering: containment
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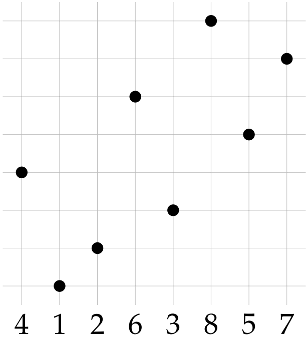
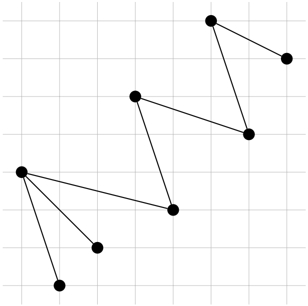
- If $\sigma \not\leq \tau$, then τ avoids σ.



Inversion graph G_π of $\pi = \pi(1) \cdots \pi(n)$:

- Vertices $= \{1, 2, \ldots, n\}$ (the same ground set)
- Edges: $a \sim b$ if $a < b$ and $b \prec a$ (edges = inversions)



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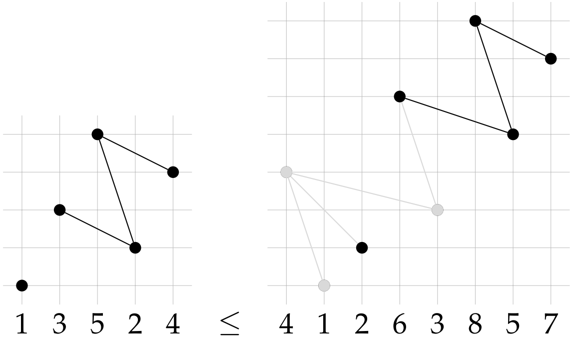

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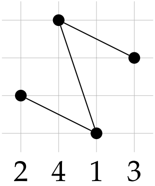
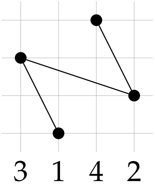
Induced substructure preserved: $\sigma \leq \pi$ implies $G_\sigma \leq_{\text{ind}} G_\pi$

Permutations to graphs is many-to-one

$\sigma \leq \pi$ implies $G_\sigma \leq_{\text{ind}} G_\pi$ but:





$G_{2413} \cong G_{3142} \cong$  even though $2413 \neq 3142$.

Define $\Sigma_\sigma = \{\text{permutations } \tau : G_\tau \cong G_\sigma\}$. ('preimage of G_σ')

Lemma

If σ satisfies $G_\sigma \leq_{ind} G_\pi$ then $\tau \leq \pi$ for some $\tau \in \Sigma_\sigma$.

Gallai (1967): characterises what's in Σ_σ.

§2 Hereditary classes and WQO

Hereditary classes

Set $\mathcal{S}$ of relational structures is a hereditary class if
$A \in \mathcal{S}$ and B is a substructure of A, then $B \in \mathcal{S}$. ('class')

Every hereditary class has a unique set of minimal forbidden structures: the smallest things that are 'not in the class'. ('basis')

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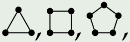
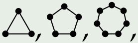
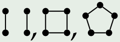
Some graph classes

Class $\mathcal{C} = \text{Free}(\mathfrak{B})$	Basis $\mathfrak{B}$
Empty graphs (no edges)	{  }
Forests	{ , ... }
Bipartite graphs	{ , ... }
Split (clique + independent)	{  }
Inversion graphs	$\text{Free}(C_{n+4}, T_2, X_2, X_3, X_{30}, X_{31}, X_{32}, X_{33}, X_{34}, X_{36}, XF_1^{2n+3}, XF_2^{n+1}, XF_3^{n}, XF_4^{n}, XF_5^{2n+3}, XF_6^{2n+2}, +\text{complements})$ (Gallai 1967)

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Some permutation classes

Class $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$	Basis $\mathfrak{B}$
$\{1, 12, 123, \dots\}$	$\{21\}$
Union of 2 increases	$\{321\}$
Union of increase & decrease	$\{3412, 2143\}$
'Stack sortable'	$\{231\}$
'2-stack-sortable'	Infinite (Murphy 2003)

Did you notice…

…that elements in the bases are pairwise incomparable?

They are antichains.

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They are antichains.

…that forests, bipartite graphs, inversion graphs and 2-stack sortable permutations have an infinite basis?

They are infinite antichains.

Infinite antichains in two places

The basis is an antichain

A class can be finitely or infinitely based.

Antichains inside the class

If a class doesn't *contain* an infinite antichain, it is well-quasi-ordered (WQO).

Motivation: tame vs wild

Finitely based classes

Structures in the class tend to be 'nice'

Can use 'basis' as input for algorithms.

WQO classes

Structures in the class tend to be 'nice'

Only countably many subclasses

Finitely based WQO classes

All of the above, plus:

Every subclass is finitely based

Graph classes

	WQO	**Not WQO**
Finitely based	Cographs Free()	Split graphs Free(, , )
Infinitely based	Linear forests Free(, , , . . .)	Forests Free(, , . . .)

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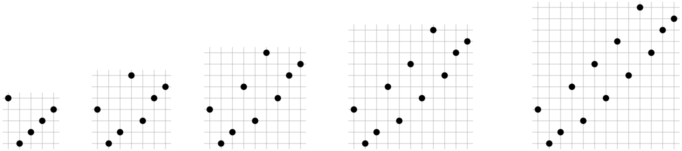
Permutation classes

	WQO	Not WQO
Finitely based	Separables Av(2413, 3142)	Increase ∪ decrease Av(2143, 3412)
Infinitely based	Av(321, 3412, 2341, 251364, $\mathfrak{Osc}$)	Av($\mathfrak{Osc}$)

Where $\mathfrak{Osc}$ is:



5 1 2 3 4   4 1 2 7 3 5 6   4 1 2 6 3 9 5 7 8   4 1 2 6 3 8 5 11 7 9 10   4 1 2 6 3 8 5 10 7 13 9 11 12

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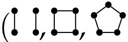
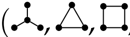
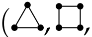
Finitely based	Cographs Free()	Split graphs Free()
Infinitely based	Linear forests Free(, . . .)	Forests Free(, . . .)

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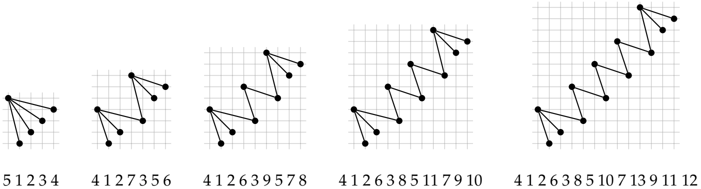
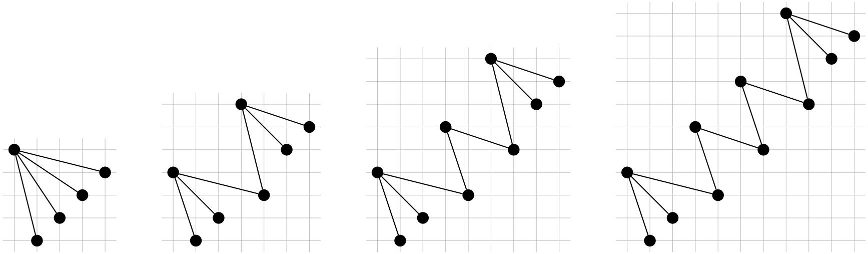


§3 Labelled WQO

A regular infinite antichain:



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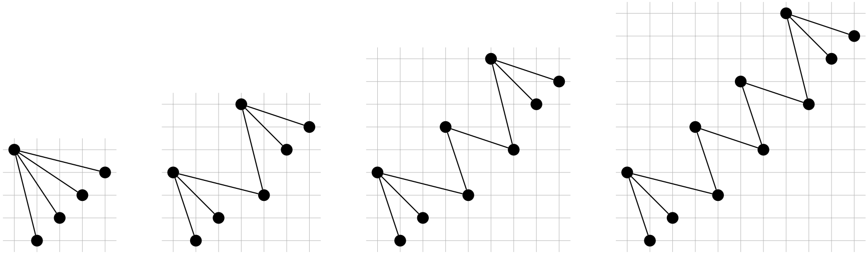

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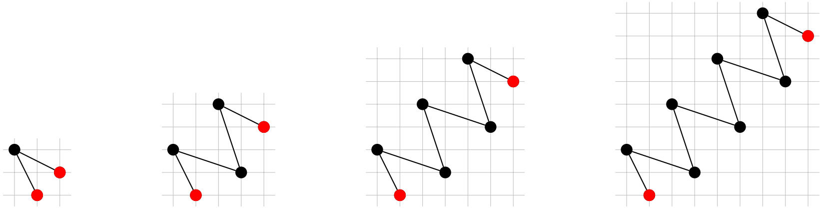


Labels can be (partially) ordered (e.g. $\bullet \preceq {\color{red}\bullet}$): embedding must respect the label ordering.

Labelled WQO

A class is labelled well-quasi-ordered (LWQO) if we cannot construct a labelled infinite antichain, no matter the set of labels.†

† Includes infinite sets of labels, but they must be WQO.

	LWQO	**Not LWQO**
Finitely based	Separables $\mathrm{Av}(2413, 3142)$	Union of 2 increases $\mathrm{Av}(321)$
Infinitely based	None	$\mathrm{Av}(321, 3412, 2341, 251364, \mathfrak{Osc})$

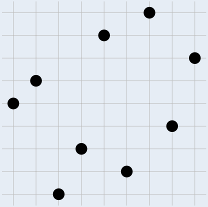
No labelled antichain $\Rightarrow$ finite basis

Proposition (After Pouzet, 1972)

An LWQO (permutation) class $\mathcal{C}$ is finitely based.

Proof.

Write $\mathcal{C} = \text{Av}(\mathfrak{B})$. For each $\beta \in \mathfrak{B}$:



β

$\square$

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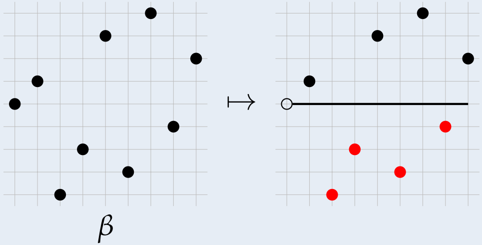


□

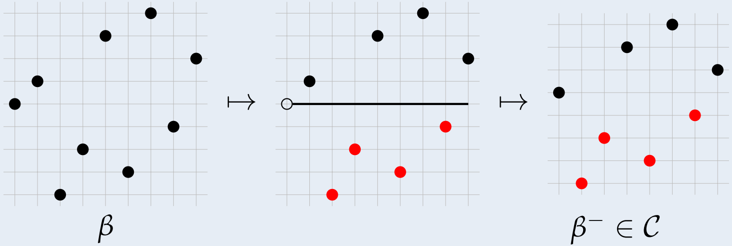
No labelled antichain ⇒ finite basis

Proposition (After Pouzet, 1972)

An LWQO (permutation) class $\mathcal{C}$ is finitely based.

Proof.

Write $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$. For each $\beta \in \mathfrak{B}$:



$\mathfrak{B}^- = \{\beta^- : \beta \in \mathfrak{B}\}$ is a labelled antichain in $\mathcal{C}$: must be finite. □

Is LWQO just WQO + finite basis?

Conjecture (Korpelainen, Lozin & Razgon, 2013)

Every finitely based WQO graph class is LWQO.

Not true for permutations:

Proposition

The class $\mathcal{C} = Av(321, 2341, 3412, 4123)$ *is WQO but not LWQO.*

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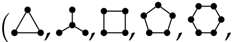
But $G_{\mathcal{C}} = \mathrm{Free}($$, \ldots)$ is not finitely based, so this is not a counterexample to the conjecture.

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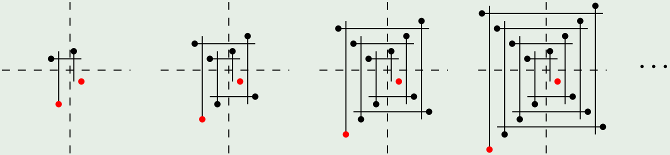
Proposition (B., Engen, Vatter, 2018)

$\mathcal{D} = Av(2143, 2413, 3412, 314562, 412563, 415632, 431562, 512364, 512643, 516432, 541263, 541632, 543162)$ *is WQO but not LWQO.*

Here's the labelled antichain



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Corollary

The class $G_{\mathcal{D}} = Free($, , , *net, co-net, rising sun, co-rising sun,* H, $\overline{H}$, *cross, co-cross,* X_{168}, $\overline{X_{168}}$, X_{160}), *is WQO but not LWQO.*

...so the conjecture is false. LWQO is *strictly* stronger than WQO + finitely based.

One-point extensions

$\mathcal{C}^{+1} = \{\pi : \text{some entry of } \pi \text{ can be removed to form } \pi^- \in \mathcal{C}\}$.

If $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$, then $\mathfrak{B} \subseteq \mathcal{C}^{+1}$.
Thus: If $\mathcal{C}^{+1}$ WQO, then $\mathcal{C}$ is finitely based (and WQO).

N.B.: $\mathcal{C}$ WQO does not imply $\mathcal{C}^{+1}$ WQO.

Lemma (Atkinson and Beals, 1999)

If $\mathcal{C}$ is finitely based, then $\mathcal{C}^{+1}$ is finitely based.

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$\mathcal{C}$ is LWQO if and only if $\mathcal{C}^{+1}$ is LWQO.

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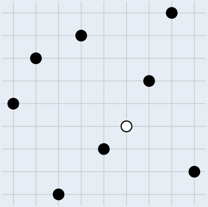
Main proof idea, showing $\mathcal{C}$ LWQO $\Rightarrow$ $\mathcal{C}^{+1}$ WQO.

For $\pi \in \mathcal{C}^{+1}$, can use 4 labels on π^- to encode extra point:



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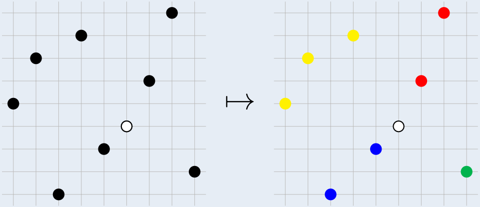

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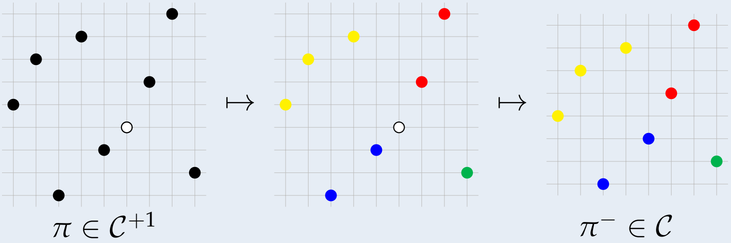

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§4 Permutations & inversion graphs

Does WQO translate?

Recall: $\sigma \leq \pi \Rightarrow G_\sigma \leq_{\text{ind}} G_\pi$. Thus $\mathcal{C}$ (L)WQO $\Rightarrow G_{\mathcal{C}}$ (L)WQO.

Question

If $\mathcal{C}$ is a permutation class such that $G_{\mathcal{C}}$ is WQO, must $\mathcal{C}$ be WQO?

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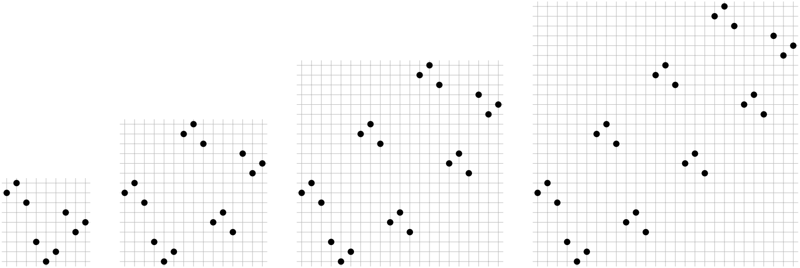

Note that $G_{231} \cong G_{312} \cong$ 

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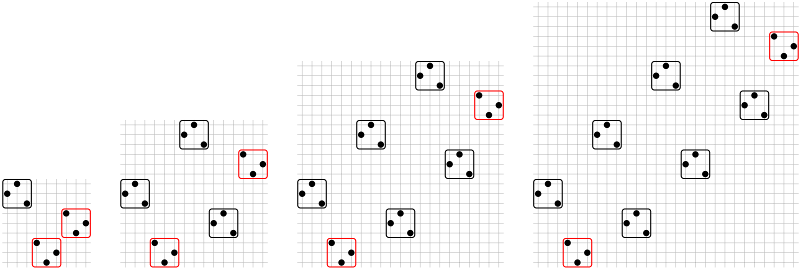

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The proof needs several ingredients:

- The substitution decomposition (a.k.a. modular decomposition)
- Nash-Williams' 1963 minimal bad sequence argument (needs Axiom of Dependent Choice)
- Gallai's 1967 characterization of

  $$\Sigma_\pi = \{\text{permutations } \sigma : G_\sigma \cong G_\pi\}.$$

  We restrict to simple permutations where $|\Sigma_\pi| \leq 4$.
- A 2019 result of Klavík and Zeman concerning automorphism groups of prime inversion graphs.

Some open questions and conjectures

Conjecture

If the permutation class $\mathcal{C}^{+1}$ is WQO, then $\mathcal{C}$ (and thus also $\mathcal{C}^{+1}$) is LWQO.

n-WQO: WQO when using a set of n incomparable labels.

Conjecture (Pouzet 1972)

A class of graphs is 2-WQO if and only if it is n-WQO for every $n \geq 2$.

Question

Is every 2-WQO permutation class also LWQO?

Thanks!