The general position problem in graphs: Some recent developments

Ullas Chandran S. V.

Mahatma Gandhi College, Thiruvananthapuram, Kerala, India

Joint with Sandi Klavžar, James Tuite and Neethu P. K.

Motivation: Dudeney's no-three-in-line problem (1917)

Dudeney's problem

This problem asks for the largest number of pawns that can be placed on a 8×8 chessboard such that **no three pawns lie on a common straight line in the plane.**

Motivation: Dudeney's no-three-in-line problem (1917)

Dudeney's problem

This problem asks for the largest number of pawns that can be placed on a 8×8 chessboard such that **no three pawns lie on a common straight line in the plane.**

The No-Three-In-Line Problem for an $n \times n$ grid is one of the oldest and most studied geometric problems.

Motivation: Dudeney's no-three-in-line problem (1917)

Dudeney's problem

This problem asks for the largest number of pawns that can be placed on a 8×8 chessboard such that **no three pawns lie on a common straight line in the plane.**

The No-Three-In-Line Problem for an $n \times n$ grid is one of the oldest and most studied geometric problems.

$2n$ is a trivial upper bound

Motivation: Dudeney's no-three-in-line problem (1917)

Dudeney's problem

This problem asks for the largest number of pawns that can be placed on a 8×8 chessboard such that **no three pawns lie on a common straight line in the plane.**

The No-Three-In-Line Problem for an $n \times n$ grid is one of the oldest and most studied geometric problems.

$2n$ is a trivial upper bound

This upper bound achieved for all n in the range $2 \leq n \leq 46$.

Motivation: Dudeney's no-three-in-line problem (1917)

Dudeney's problem

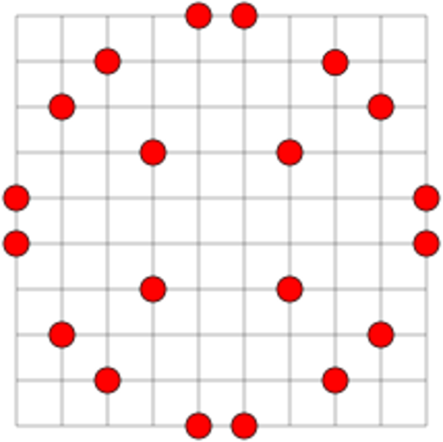
This problem asks for the largest number of pawns that can be placed on a 8×8 chessboard such that **no three pawns lie on a common straight line in the plane.**

The No-Three-In-Line Problem for an $n \times n$ grid is one of the oldest and most studied geometric problems.

$2n$ is a trivial upper bound

This upper bound achieved for all n in the range $2 \leq n \leq 46$.

Conjecture: For large values of n, the number of points is $\leq 1.814n$



A set of 20 points in a 10 x 10 grid, with no three points in a line (*source: Wikipedia*)

4

The general position problem in graph theory

The general position problem in graph theory

This is a natural extension of Dudeney's No-Three-In-Line problem to graphs.

The general position problem in graph theory

This is a natural extension of Dudeney's No-Three-In-Line problem to graphs.

Interested in studying sets of vertices of a graph G which have a no-three-in-line property.

The general position problem in graph theory

This is a natural extension of Dudeney's No-Three-In-Line problem to graphs.

Interested in studying sets of vertices of a graph G which have a no-three-in-line property.

what we mean by a 'line' in a graph?

The general position problem in graph theory

This is a natural extension of Dudeney's No-Three-In-Line problem to graphs.

Interested in studying sets of vertices of a graph G which have a no-three-in-line property.

what we mean by a 'line' in a graph?

The 'classical' definition of the general position problem for graphs: 'line' to mean 'shortest path.'

The general position problem in graph theory

This is a natural extension of Dudeney's No-Three-In-Line problem to graphs.

Interested in studying sets of vertices of a graph G which have a no-three-in-line property.

what we mean by a 'line' in a graph?

The 'classical' definition of the general position problem for graphs: 'line' to mean 'shortest path.'

This problem was introduced in graphs independently by Klavžar & Manuel in 2018 and Ullas Chandran & Parthasarathy in 2016 (under the name geodesic irredundant set)

Definition 1

- A subset $S \subseteq V(G)$ is in *general position* if no three vertices of S are contained in a common shortest path of G.

Definition 1

- A subset $S \subseteq V(G)$ is in *general position* if no three vertices of S are contained in a common shortest path of G.

- The *general position number* $\mathrm{gp}(G)$ is the number of vertices in a largest general position set of G.

Definition 1

- A subset $S \subseteq V(G)$ is in *general position* if no three vertices of S are contained in a common shortest path of G.
- The *general position number* $\mathrm{gp}(G)$ is the number of vertices in a largest general position set of G.
- A largest general position set of G is also called a gp-*set*.

Some simple examples

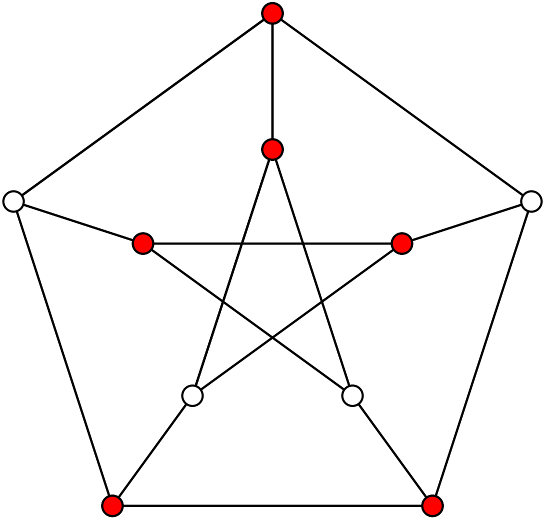


Figure: A gp-set of the Petersen graph

Some simple examples and observation

- $\max\{2, \omega(G)\} \leq \text{gp}(G) \leq n(G)$, where $\omega(G) \to$ clique number of G

Some simple examples and observation

- $\max\{2, \omega(G)\} \leq \text{gp}(G) \leq n(G)$, where $\omega(G) \rightarrow$ clique number of G
- $\text{gp}(P_n) = 2 (n \geq 2)$; and $\text{gp}(C_n) = 3 (n \geq 5)$

Some simple examples and observation

- $\max\{2, \omega(G)\} \leq \text{gp}(G) \leq n(G)$, where $\omega(G) \rightarrow$ clique number of G
- $\text{gp}(P_n) = 2 (n \geq 2)$; and $\text{gp}(C_n) = 3 (n \geq 5)$
- $\text{gp}(G) = n(G)$ if and only if $G = K_n$

Some simple examples and observation

- $\max\{2, \omega(G)\} \leq \text{gp}(G) \leq n(G)$, where $\omega(G) \to$ clique number of G
- $\text{gp}(P_n) = 2 (n \geq 2)$; and $\text{gp}(C_n) = 3 (n \geq 5)$
- $\text{gp}(G) = n(G)$ if and only if $G = K_n$
- $\text{gp}(K_{m,n}) = m$ if $m \geq n \geq 2$

Some simple examples and observation

- $\max\{2, \omega(G)\} \leq gp(G) \leq n(G)$, where $\omega(G) \rightarrow$ clique number of G
- $gp(P_n) = 2 (n \geq 2)$; and $gp(C_n) = 3 (n \geq 5)$
- $gp(G) = n(G)$ if and only if $G = K_n$
- $gp(K_{m,n}) = m$ if $m \geq n \geq 2$

block graphs

- A vertex v is a *simplicial vertex* in a graph G if the subgraph induced by its neighbors forms a clique

block graphs

- A vertex v is a *simplicial vertex* in a graph G if the subgraph induced by its neighbors forms a clique
- The set of all simplicial vertices of a block graph G is a maximum general position set of G [Paul Manuel, Sandi Klavžar, 2018].

block graphs

- A vertex v is a *simplicial vertex* in a graph G if the subgraph induced by its neighbors forms a clique
- The set of all simplicial vertices of a block graph G is a maximum general position set of G [Paul Manuel, Sandi Klavžar, 2018].
- In a tree T, the general position number is the number of end vertices of T [Paul Manuel, Sandi Klavžar, 2018].

Graphs with small or large general position numbers

- $\mathrm{gp}(G) \leq n(G) - \mathrm{diam}(G) + 1$ [Ullas Chandran, Parthasarathy, 2016]

Graphs with small or large general position numbers

- $\mathrm{gp}(G) \leq n(G) - \mathrm{diam}(G) + 1$ [Ullas Chandran, Parthasarathy, 2016]
- $\mathrm{gp}(G) = 2$ if and only if $G \in \{C_4, P_n\}$ [Ullas Chandran, Parthasarathy, 2016]

Graphs with small or large general position numbers

- $\text{gp}(G) \leq n(G) - \mathrm{diam}(G) + 1$ [Ullas Chandran, Parthasarathy, 2016]
- $\text{gp}(G) = 2$ if and only if $G \in \{C_4, P_n\}$ [Ullas Chandran, Parthasarathy, 2016]
- Graphs with $\text{gp}(G) = n(G) - 1$ characterized [Ullas Chandran, Parthasarathy, 2016]

Graphs with small or large general position numbers

- $\mathrm{gp}(G) \leq n(G) - \mathrm{diam}(G) + 1$ [Ullas Chandran, Parthasarathy, 2016]
- $\mathrm{gp}(G) = 2$ if and only if $G \in \{C_4, P_n\}$ [Ullas Chandran, Parthasarathy, 2016]
- Graphs with $\mathrm{gp}(G) = n(G) - 1$ characterized [Ullas Chandran, Parthasarathy, 2016]
- Graphs with $\mathrm{gp}(G) = n(G) - 2$ characterized [Elias Thomas, Ullas Chandran, 2021]

Characterization of general position sets

- $X \subseteq V(G)$, and $\mathcal{P} = \{X_1, \ldots, X_p\}$ a partition of X.
- $\mathcal{P}$ is *distance-constant* if the distance $d_G(u, v)$, $u \in X_i$, $v \in X_j$, is independent of the selection of u and v.

Characterization of general position sets

- $X \subseteq V(G)$, and $\mathcal{P} = \{X_1, \ldots, X_p\}$ a partition of X.
- $\mathcal{P}$ is *distance-constant* if the distance $d_G(u, v)$, $u \in X_i$, $v \in X_j$, is independent of the selection of u and v.
- For $i \neq j$, then the distance $d_G(X_i, X_j)$ can be defined as the distance between a vertex of X_i and a vertex of X_j.

Characterization of general position sets

- $X \subseteq V(G)$, and $\mathcal{P} = \{X_1, \ldots, X_p\}$ a partition of X.
- $\mathcal{P}$ is *distance-constant* if the distance $d_G(u, v)$, $u \in X_i$, $v \in X_j$, is independent of the selection of u and v.
- For $i \neq j$, then the distance $d_G(X_i, X_j)$ can be defined as the distance between a vertex of X_i and a vertex of X_j.
- A distance-constant partition $\mathcal{P}$ is *in-transitive* if $d_G(X_i, X_k) \neq d_G(X_i, X_j) + d_G(X_j, X_k)$ holds.

Characterization of general position sets

- $X \subseteq V(G)$, and $\mathcal{P} = \{X_1, \ldots, X_p\}$ a partition of X.
- $\mathcal{P}$ is *distance-constant* if the distance $d_G(u, v)$, $u \in X_i$, $v \in X_j$, is independent of the selection of u and v.
- For $i \neq j$, then the distance $d_G(X_i, X_j)$ can be defined as the distance between a vertex of X_i and a vertex of X_j.
- A distance-constant partition $\mathcal{P}$ is *in-transitive* if $d_G(X_i, X_k) \neq d_G(X_i, X_j) + d_G(X_j, X_k)$ holds.

Theorem 2 (Anand et al., 2019)

Then $X \subseteq V(G)$ is a general position set if and only if the components of $G[X]$ are cliques, the vertices of which form an in-transitive, distance-constant partition of X.

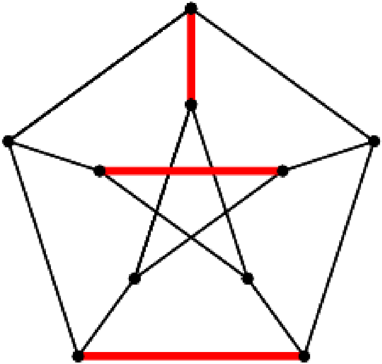


Figure: Peterson graph

The general position number was studied in many classes of graphs and products

- Direct products [K. Yero, 2019]
- Generalized lexicographic products [K. Yero, 2019]
- Join and coronas [Ghorbani et al, 2021]
- Line graphs of complete graphs[Ghorbani et al, 2021]
- Cartesian products [Klavzar et al, Tian, 2021]
- Complementary prisms [Neethu et al., 2021]
- Extremal position problems [James Tuite et al., 2022, 2024]
- Mycielskian graphs [Thomas et al., 2024]
- Double graphs [Dhanya Roy et al., 2024]

The general position number was studied in many classes of graphs and products

- Direct products [K. Yero, 2019]
- Generalized lexicographic products [K. Yero, 2019]
- Join and coronas [Ghorbani et al, 2021]
- Line graphs of complete graphs[Ghorbani et al, 2021]
- Cartesian products [Klavzar et al, Tian, 2021]
- Complementary prisms [Neethu et al., 2021]
- Extremal position problems [James Tuite et al., 2022, 2024]
- Mycielskian graphs [Thomas et al., 2024]
- Double graphs [Dhanya Roy et al., 2024]

U. Chandran S.V., S. Klavžar, J. Tuite, The general position problem: a survey, arXiv:2501.19385 (2025).

Monophonic position problem

It obtained by replacing 'shortest path' by 'induced path' in the general position problem.

Monophonic position problem

It obtained by replacing 'shortest path' by 'induced path' in the general position problem.

Definition 3

A set $M \subseteq V(G)$ is a *monophonic position set* or *mp-set* of G if no three vertices of M lie on a common monophonic path in G. The *monophonic position number* or *mp-number* $\text{mp}(G)$ of G is the number of vertices in a largest mp-set of G.

Monophonic position problem

It obtained by replacing 'shortest path' by 'induced path' in the general position problem.

Definition 3

A set $M \subseteq V(G)$ is a *monophonic position set* or *mp-set* of G if no three vertices of M lie on a common monophonic path in G. The *monophonic position number* or *mp-number* $\text{mp}(G)$ of G is the number of vertices in a largest mp-set of G.

E.J. Thomas, U. Chandran S.V., J. Tuite, G. Di Stefano, On monophonic position sets in graphs, Discrete Appl. Math. 354 (2024) 72–82.

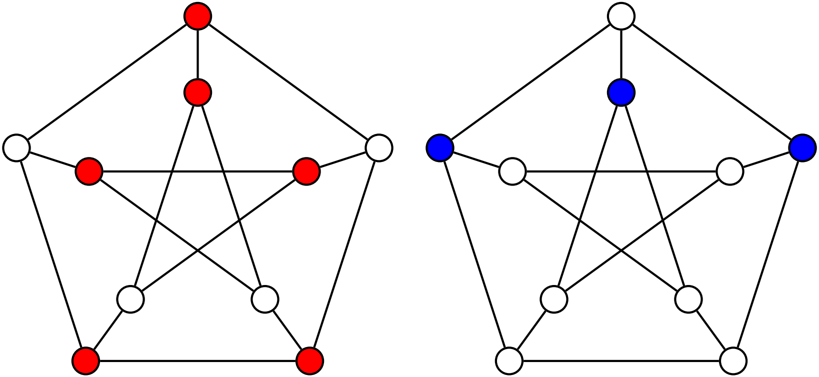


Figure: The Petersen graph with a maximum gp-set (left) and a maximum mp-set (right)

- $\max\{2, \omega(G)\} \leq \mathrm{mp}(G) \leq \mathrm{gp}(G)$

- $\max\{2, \omega(G)\} \leq \mathrm{mp}(G) \leq \mathrm{gp}(G)$
- $\mathrm{mp}(G) = n$ if and only if $G \cong K_n$

- $\max\{2, \omega(G)\} \leq \text{mp}(G) \leq \text{gp}(G)$
- $\text{mp}(G) = n$ if and only if $G \cong K_n$
- $\text{mp}(P_n) = 2$ for $n \geq 2$.

- $\max\{2, \omega(G)\} \leq \mathrm{mp}(G) \leq \mathrm{gp}(G)$
- $\mathrm{mp}(G) = n$ if and only if $G \cong K_n$
- $\mathrm{mp}(P_n) = 2$ for $n \geq 2$.
- $\mathrm{mp}(C_n) = 2$ for $n \geq 4$ (Recall that $\mathrm{gp}(C_n) = 3$ for $n \geq 5$)

- $\max\{2, \omega(G)\} \leq \text{mp}(G) \leq \text{gp}(G)$
- $\text{mp}(G) = n$ if and only if $G \cong K_n$
- $\text{mp}(P_n) = 2$ for $n \geq 2$.
- $\text{mp}(C_n) = 2$ for $n \geq 4$ (Recall that $\text{gp}(C_n) = 3$ for $n \geq 5$)
- $\text{gp}(G)$ and $\text{mp}(G)$ coincide for distance-hereditary graphs (in particular for cographs, block graphs and trees).

Some simple observations

- $\text{mp}(G) \leq n - c(G)$, where $c(G) \rightarrow$ the number of cut-vertices of G

Some simple observations

- $\text{mp}(G) \leq n - c(G)$, where $c(G) \rightarrow$ the number of cut-vertices of G
- $\text{mp}(G) \leq n - L + 1$, where L denote the longest induced path in G.

Some simple observations

- $\text{mp}(G) \leq n - c(G)$, where $c(G) \rightarrow$ the number of cut-vertices of G
- $\text{mp}(G) \leq n - L + 1$, where L denote the longest induced path in G.
- $\text{mp}(G) \geq s(G)$, where $s(G)$ denote the number of simplicial vertices of G

Some simple observations

- $\text{mp}(G) \leq n - c(G)$, where $c(G) \rightarrow$ the number of cut-vertices of G
- $\text{mp}(G) \leq n - L + 1$, where L denote the longest induced path in G.
- $\text{mp}(G) \geq s(G)$, where $s(G)$ denote the number of simplicial vertices of G

Lemma 4 (Thomas et al., 2024)

Let G be a connected graph and $M \subseteq V(G)$ be an mp-set. Then $G[M]$ is a disjoint union of k cliques $G[M] = \bigcup_{i=1}^{k} W_i$. If $k \geq 2$, then for $1 \leq i \leq k$ any two vertices of W_i have a common neighbour in $G \setminus M$.

Triangle-free graphs

Theorem 5 (Thomas et al., 2024)

The mp-number of a connected triangle-free graph G with order $n \geq 3$ satisfies $\text{mp}(G) \leq \alpha(G)$. *Moreover, if the length of any monophonic path is at most three, then* $\text{mp}(G) = \alpha(G)$.

Corona products

Theorem 6 (Thomas et al., 2024)

Let G be a connected graph with order $n(G)$ and let H be any graph. Then $\mathrm{mp}(G \odot H) = n(G)\,\mathrm{mp}(H)$.

Join of graphs

Theorem 7 (Thomas et al., 2024)

The monophonic position number of the join $G \vee H$ of graphs G and H is related to the monophonic position numbers of G and H by

$$\mathrm{mp}(G \vee H) = \max\{\omega(G) + \omega(H), \mathrm{mp}(G), \mathrm{mp}(H)\}.$$

mp(G) and gp(G) can be far apart

Theorem 8 (Thomas et al., 2024)

For any $2 \leq a \leq b$ there exists a graph G with $\mathrm{mp}(G) = a$ and $\mathrm{gp}(G) = b$.

Complexity

Definition 9

GENERAL POSITION SET
INSTANCE: A graph G, a positive integer $k \leq |V(G)|$.
QUESTION: Is there a general position set S for G such that $|S| \geq k$?

Theorem 10 (Manuel and Klavžar, 2018+)

GENERAL POSITION SET *is NP-Complete.*

The proof of this is based on a reduction from the NP-complete MAXIMUM INDEPENDENT SET problem.

Complexity

Definition 11

MONOPHONIC POSITION SET
INSTANCE: A graph G, a positive integer $k \leq |V(G)|$.
QUESTION: Is there a monophonic position set S for G such that $|S| \geq k$?

Theorem 12 (Thomas et al., 2024)

The MONOPHONIC POSITION SET *problem is NP-hard.*

The proof of this is based on a reduction from the NP-complete CLIQUE problem

It is not clear whether MONOPHONIC POSITION SET is NP-complete for general graphs, since to this end we would also have to prove that a solution can be verified in polynomial time.

Proof

- We polynomially transform an instance (G, k) of CLIQUE to an instance (G', k') of MONOPHONIC POSITION SET
- Hence G has a clique of order k or more if and only if G' has a monophonic position set of order k' or more.
- Given an instance (G, k), the graph G' is built as follows: it contains a subgraph H isomorphic to G and a clique graph H' of order $n = |V(G)|$ such that $G' = H \vee H'$.
- For k', we set $k' = n + k$
- $\omega(G') = \omega(G) + n$
- Then $\text{mp}(G') = \text{mp}(H \vee H') = \max\{\omega(H) + \omega(H'), \text{mp}(H), \text{mp}(H')\}$
- $\text{mp}(G') = \max\{\omega(H) + n, \text{mp}(H), n\} = \omega(H) + n$
- Then $\omega(G) = \omega(H) \geq k$ iff $\text{mp}(G') = \omega(H) + n \geq k + n = k'$

Some simple but challenging open problems

- Characterize graphs for which $\mathrm{gp}(G) = 3$

Some simple but challenging open problems

- Characterize graphs for which $\mathrm{gp}(G) = 3$
- Characterize graphs for which $\mathrm{gp}(G) = \omega(G)$

Some simple but challenging open problems

- Characterize graphs for which $\mathrm{gp}(G) = 3$
- Characterize graphs for which $\mathrm{gp}(G) = \omega(G)$
- Characterize graphs for which $\mathrm{gp}(G) = \mathrm{mp}(G)$

Some simple but challenging open problems

- Characterize graphs for which $\mathrm{gp}(G) = 3$
- Characterize graphs for which $\mathrm{gp}(G) = \omega(G)$
- Characterize graphs for which $\mathrm{gp}(G) = \mathrm{mp}(G)$
- Characterize graphs for which $\mathrm{mp}(G) = 2$ or 3

Some simple but challenging open problems

- Characterize graphs for which $\mathrm{gp}(G) = 3$
- Characterize graphs for which $\mathrm{gp}(G) = \omega(G)$
- Characterize graphs for which $\mathrm{gp}(G) = \mathrm{mp}(G)$
- Characterize graphs for which $\mathrm{mp}(G) = 2$ or 3
- Characterize graphs for which $\mathrm{mp}(G) = n(G) - 2$

Some simple but challenging open problems

- Characterize graphs for which $\mathrm{gp}(G) = 3$
- Characterize graphs for which $\mathrm{gp}(G) = \omega(G)$
- Characterize graphs for which $\mathrm{gp}(G) = \mathrm{mp}(G)$
- Characterize graphs for which $\mathrm{mp}(G) = 2$ or 3
- Characterize graphs for which $\mathrm{mp}(G) = n(G) - 2$
- Characterize graphs for which $\mathrm{mp}(G) = \omega(G)$

The mp-number of Cartesian & lexicographic products

Results are taken from:
U. Chandran S.V., S. Klavžar, P.K. Neethu, J. Tuite, On monophonic position sets of Cartesian and lexicographic products of graphs, arXiv:2412.09837v2 [math.CO], 2025

The mp-number of Cartesian & lexicographic products

Results are taken from:
U. Chandran S.V., S. Klavžar, P.K. Neethu, J. Tuite, On monophonic position sets of Cartesian and lexicographic products of graphs, arXiv:2412.09837v2 [math.CO], 2025

Both products have vertex set $V(G) \times V(H)$.

The mp-number of Cartesian & lexicographic products

Results are taken from:
U. Chandran S.V., S. Klavžar, P.K. Neethu, J. Tuite, On monophonic position sets of Cartesian and lexicographic products of graphs, arXiv:2412.09837v2 [math.CO], 2025

Both products have vertex set $V(G) \times V(H)$.

In the Cartesian product $G \,\square\, H$ the vertices (g_1, h_1) and (g_2, h_2) are adjacent if (i) $g_1 \sim g_2$ in G and $h_1 = h_2$ or (ii) $g_1 = g_2$ and $h_1 \sim h_2$ in H.

The mp-number of Cartesian & lexicographic products

Results are taken from:
U. Chandran S.V., S. Klavžar, P.K. Neethu, J. Tuite, On monophonic position sets of Cartesian and lexicographic products of graphs, arXiv:2412.09837v2 [math.CO], 2025

Both products have vertex set $V(G) \times V(H)$.

In the Cartesian product $G \,\square\, H$ the vertices (g_1, h_1) and (g_2, h_2) are adjacent if (i) $g_1 \sim g_2$ in G and $h_1 = h_2$ or (ii) $g_1 = g_2$ and $h_1 \sim h_2$ in H.

In the lexicographic product $G \circ H$ these vertices are adjacent if (i) $g_1 \sim g_2$ or (ii) $g_1 = g_2$ and $h_1 \sim h_2$.

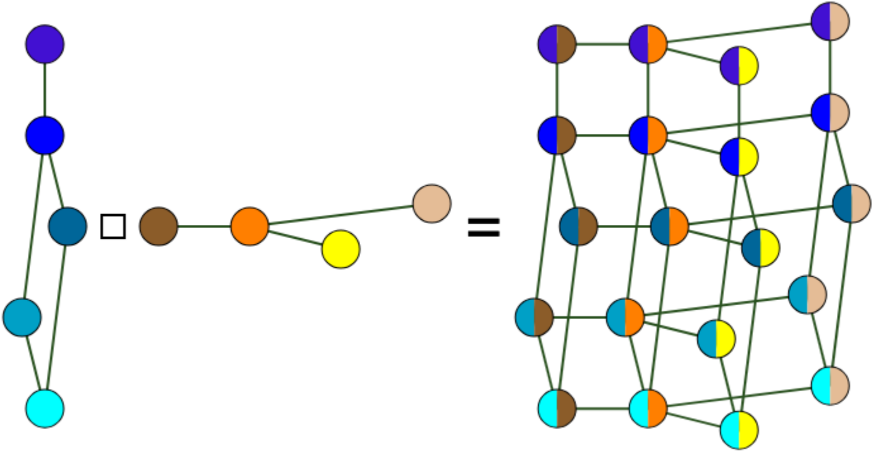


Figure: Cartesian product of graphs(source: Wikipedia)

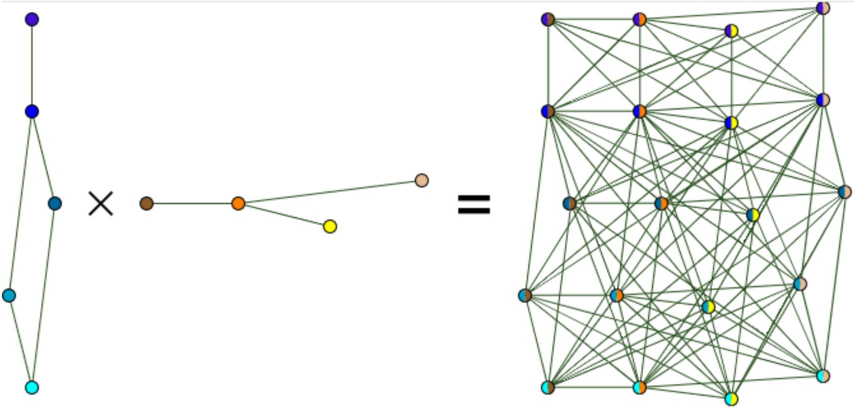


Figure: Lexicographic product of graphs(source: Wikipedia)

If $S \subseteq V(G \,\Box\, H)$, then the set $\{g \in V(G) : \ (g, h) \in S$ for some $h \in V(H)\}$ is the *projection* $\pi_G(S)$ of S on G.

If $S \subseteq V(G \,\square\, H)$, then the set $\{g \in V(G) : (g, h) \in S \text{ for some } h \in V(H)\}$ is the *projection* $\pi_G(S)$ of S on G.

If $v_0, v_1, \ldots, v_{\ell-1}, v_\ell$ is a path Q in H, then by ${}_uQ$ we denote the path

$$(u, v_0), (u, v_1), \ldots, (u, v_{\ell-1}), (u, v_\ell)$$

in $G * H$.

Cartesian product

Observation: For any graphs G and H, $\text{mp}(G \,\square\, H) \geq \max\{\omega(G), \omega(H)\}$.

The observation follows from the fact that for any graph G we have $\text{mp}(G) \geq \omega(G)$, and the clique number of the Cartesian product is given by $\omega(G \,\square\, H) = \max\{\omega(G), \omega(H)\}$.

Main result

A vertex subset S of a graph that is simultaneously an independent set and in monophonic position is called an *independent monophonic position set*. The largest order of an independent monophonic position set is the *independent monophonic position number* of G $\mathrm{mp}_i(G)$. Also, for any graph $\sigma(G) = 1$ if G contains a simplicial vertex and $\sigma(G) = 0$ otherwise.

Theorem 13

If G and H are connected graphs, then

$$\mathrm{mp}(G \,\square\, H) \leq \max\{\omega(G), \omega(H), \sigma(G)\,\mathrm{mp}_i(H), \sigma(H)\,\mathrm{mp}_i(G)\}.$$

Furthermore, if neither G nor H has simplicial vertices, then

$$\mathrm{mp}(G \,\square\, H) = \max\{\omega(G), \omega(H)\}.$$

Main result

A vertex subset S of a graph that is simultaneously an independent set and in monophonic position is called an *independent monophonic position set*. The largest order of an independent monophonic position set is the *independent monophonic position number* of G $\mathrm{mp}_i(G)$. Also, for any graph $\sigma(G) = 1$ if G contains a simplicial vertex and $\sigma(G) = 0$ otherwise.

Theorem 13

If G and H are connected graphs, then

$$\mathrm{mp}(G \,\square\, H) \leq \max\{\omega(G), \omega(H), \sigma(G)\,\mathrm{mp}_i(H), \sigma(H)\,\mathrm{mp}_i(G)\}.$$

Furthermore, if neither G nor H has simplicial vertices, then

$$\mathrm{mp}(G \,\square\, H) = \max\{\omega(G), \omega(H)\}\,.$$

To prove this result, we need a sequence of lemmas.

Structure of mp-sets

Lemma 14

If S is a monophonic position set of $G \,\square\, H$, then $\pi_H(S)$ is a monophonic position set of H.

Structure of mp-sets

Lemma 14

If S is a monophonic position set of $G \,\square\, H$, then $\pi_H(S)$ is a monophonic position set of H.

- If $\pi_H(S)$ is not in monophonic position in H. Then there exists a set $S' = \{v_1, v_2, v_3\} \subseteq \pi_H(S)$ such that there is an induced v_1, v_3-path P in H that passes through v_2.
- Since $\{v_1, v_2, v_3\} \subseteq \pi_H(S)$, there exist vertices u_1, u_2, u_3 of G such that (u_1, v_1) (u_2, v_2) and (u_3, v_3) belong to S;
- we will derive a contradiction by constructing a monophonic path in $G \,\square\, H$ from (u_1, v_1) to (u_3, v_3) that passes through (u_2, v_2).
- Let Q and R be monophonic paths in G from u_1 to u_2 and from u_2 to u_3 respectively.

Without loss of generality, there are four possibilities to consider:

Case	Section 1	Section 2	Section 3
u_1, u_2, u_3 distinct	Q_{v_1}	${}_{u_2}P$	R_{v_3}
$u_1 = u_2 \neq u_3$	${}_{u_1}P$	R_{v_3}	
$u_1 = u_3 \neq u_2$	Q_{v_1}	${}_{u_2}P$	$\widetilde{Q}_{v_3}$
$u_1 = u_2 = u_3$	${}_{u_1}P$		

Table: Construction of S-bad paths

The desired monophonic paths in $G \,\square\, H$ are constructed by concatenating the paths in Table 1 in order.

Lemma 15

Let G and H be connected graphs and S be a monophonic position set of $G \,\Box\, H$. If $(u,v) \in S$, then $V({}^{u}H) \cap S = \{(u,v)\}$ or $V(G^{v}) \cap S = \{(u,v)\}$.

Lemma 15

Let G and H be connected graphs and S be a monophonic position set of $G \,\Box\, H$. If $(u, v) \in S$, then $V({}^{u}H) \cap S = \{(u, v)\}$ or $V(G^{v}) \cap S = \{(u, v)\}$.

- Suppose that the result is not true, i.e. that there exist $u' \neq u$ in G and $v' \neq v$ in H such that (u, v), (u', v) and (u, v') all belong to S.
- Let P be a monophonic u', u-path in G and Q be a monophonic v, v'-path in H.
- Then the concatenation of P_v and ${}_{u}Q$ would be a monophonic $(u', v), (u, v')$-path in $G \,\Box\, H$ passing through (u, v), a contradiction.

Lemma 16

If $S = \{(u_i, v_i) : i \in [r]\}$ *is a monophonic position set of* $G \,\Box\, H$ *for some* $r \geq 2$*, then one of the following holds:*

(a) S *lies in a single* G*-layer, or* S *lies in a single* H*-layer,*

(b) $u_1, \ldots, u_r$ *are distinct vertices of* G *and* $v_1, \ldots, v_r$ *are distinct vertices of* H*, and neither of these sets induce a clique, or*

(c) $\pi_G(S)$ *is a clique of* G *with order at least* 2 *and* $v_1, \ldots, v_r$ *are distinct vertices of* H*, or* $\pi_H(S)$ *is a clique of* H *with order at least* 2 *and* $u_1, \ldots, u_r$ *are distinct vertices of* G*.*

Definition 17

We will call a monophonic position set of Type (a) *layered*, of Type (b) *varied*, and of Type (c) *cliquey*. These three types of monophonic position sets are shown schematically in the following.

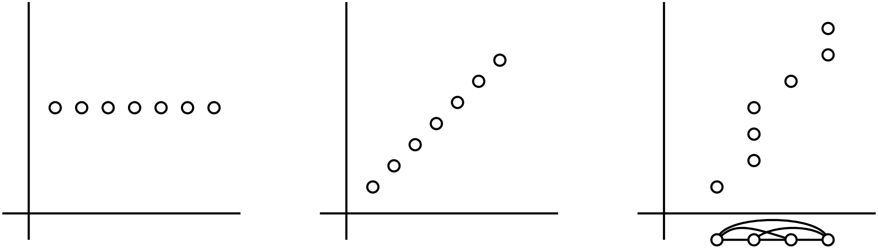


Figure: Layered (left), varied (middle), and cliquey (right) monophonic position sets

Immediate consequences

Corollary 18

If G and H are connected graphs, then

$$\mathrm{mp}(G \,\square\, H) \leq \max\{\mathrm{mp}(G), \mathrm{mp}(H)\}\,.$$

$|\pi_G(S)| = |S|$ or $|\pi_H(S)| = |S|$ (or both), and so the conclusion follows

Corollary 19

For paths P_m, P_n of order at least two and cycles C_r, C_s of length at least four,

$$\mathrm{mp}(P_m \,\square\, P_n) = \mathrm{mp}(P_n \,\square\, C_r) = \mathrm{mp}(C_r \,\square\, C_s) = 2\,.$$

Corollary 20

If H is a connected graph and $n \geq \mathrm{mp}(H)$, *then* $\mathrm{mp}(K_n \,\Box\, H) = n$.

Corollary 20

If H is a connected graph and $n \geq \mathrm{mp}(H)$, then $\mathrm{mp}(K_n \,\square\, H) = n$.

This yields $\mathrm{mp}(K_n \,\square\, P_m) = n$ for $n \geq 2, m \geq 4$; $\mathrm{mp}(K_n \,\square\, C_m) = n$ for $n \geq 3$; and $\mathrm{mp}(K_n \,\square\, K_m) = \max\{n, m\}$.

Varied sets

We show that varied monophonic position sets can only contain at most two vertices.

Lemma 21

If $u, u' \in V(G)$ and $v, v' \in V(H)$ are such that $u' \notin N_G[u]$ and $v' \notin N_H[v]$, then the set $\{(u, v), (u', v')\}$ is a maximal monophonic position set of $G \,\square\, H$.

Lemma 22

If uu' is an edge of G and vv' an edge of H, then $\{(u, v), (u', v')\}$ is a maximal monophonic position set of $G \,\square\, H$.

Cliquey and layered sets

- We now restrict our attention to layered and cliquey monophonic position sets.
- w.l.g, we will assume that layered sets lie in a H-layer $\{u\} \times V(H)$ and that if S is cliquey, then $\pi_G(S)$ is a clique of order at least two.
- We will use the following labelling convention for layered and cliquey monophonic position sets S of $G \,\square\, H$: with each vertex $u_i \in \pi_G(S)$ we associate the set $S_i' = \pi_H(S \cap (\{u_i\} \times V(H))$.

  Recall that by Lemma 16 the sets S_i' are pairwise disjoint, so that these sets partition $\pi_H(S)$.
  The sets S_i' are schematically represented in the following figure.

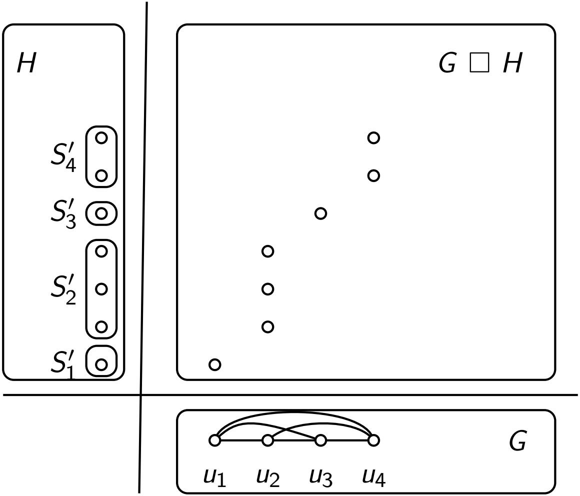


Figure: Sets S_i'

Lemma 23

If $\{u\} \times S' \subseteq S$, where $S' \subseteq V(H)$, then if S is either cliquey or S is layered with $|S| > \max\{\omega(G), \omega(H)\}$, then S' induces an independent set in H, with a similar result for subsets $S'' \times \{v\} \subseteq S$.

Lemma 24

Let S be a cliquey monophonic position set of $G \,\square\, H$, where $\pi_G(S)$ is a clique. Then $\pi_H(S)$ is an independent set.

Lemma 25

If S is a layered or cliquey monophonic position set of $G \,\square\, H$ (where $\pi_G(S)$ is a clique) and $|S| > \max\{\omega(G), \omega(H)\}$, then every vertex from $\pi_G(S)$ is simplicial.

Main result

Theorem 26

If G and H are connected graphs, then

$$\mathrm{mp}(G \,\square\, H) \leq \max\{\omega(G), \omega(H), \sigma(G)\,\mathrm{mp}_i(H), \sigma(H)\,\mathrm{mp}_i(G)\}.$$

Furthermore, if neither G nor H has simplicial vertices, then

$$\mathrm{mp}(G \,\square\, H) = \max\{\omega(G), \omega(H)\}\,.$$

Proof

- Suppose that $G \,\square\, H$ contains a monophonic position set S with $|S| > \max\{\omega(G), \omega(H)\}$.
- Then it follows from Lemmas 21 and 22 that S is either cliquey or layered.
- We assume that $\pi_G(S)$ is a clique. Then by Lemma 25 it is necessary that every vertex of $\pi_G(S)$ is simplicial, so that $\sigma(G) = 1$.
- Lemmas 14 and 24 show that $\pi_H(S)$ is an independent monophonic position set of H and hence $\text{mp}(G \,\square\, H) = |S| = |\pi_H(S)| \leq \text{mp}_i(H)$.

Lomer bound

For a vertex of any graph G we will write $\ell(u)$ for the number of neighbours of u that are leaves, and set $\Delta_1(G) = \max\{\ell(u) : u \in V(G)\}$.

Theorem 27

If G and H are graphs with order at least three that both contain leaves, then

$$\text{mp}(G \,\square\, H) \geq \max\{\Delta_1(G), \Delta_1(H)\}.$$

$\text{mp}(K_{1,n} \,\square\, K_{1,m}) = n = \text{mp}_i(K_{1,n})$. This demonstrates that the upper bound in Theorem 26 and the lower bound in Proposition 27 are sharp.

The structural properties established can be applied to triangle-free graphs.

$\sigma(G) = 1$ if G contains a leaf and $\sigma(G) = 0$ otherwise.

Theorem 28

If G and H are connected triangle-free graphs of order at least 3, then

$$\text{mp}(G \,\square\, H) \leq \max\{2, \sigma(G)\Delta(H), \sigma(H)\Delta(G)\}\,.$$

Lexicographic products

We recall the distance function of lexicographic products and state two lemmas.
If (g,h) and (g',h') are distinct vertices of $G \circ H$, then

$$d_{G\circ H}\left((g,h),(g',h')\right) = \begin{cases} d_G(g,g'); & g \neq g'\,, \\ d_H(h,h'); & g = g', \deg_G(g) = 0\,, \\ \min\{d_H(h,h'),2\}; & g = g', \deg_G(g) \neq 0\,. \end{cases}$$

Lemma 29

If S is a monophonic position set of $G \circ H$, then $\pi_G(S)$ is a monophonic position set of G.

Let $u \in V(G)$ and and let P be a monophonic path in H. Then the isomorphic copy of P in the layer ${}^uH$ of $G \circ H$ is a monophonic path of $G \circ H$. This fact implies the following lemma.

Lemma 30

If S is a monophonic position set of $G \circ H$, then for any $u \in \pi_G(S)$, $\pi_H({}^uH \cap S)$ is a monophonic position set of H.

Main result

For a monophonic position set M of G we denote the components of $G[M]$ by: $A_1, A_2, \ldots, A_k, B_1, \ldots, B_r$, where $|A_i| \geq 2$ for each $i \in [k]$ and $|B_j| = 1$ for each $j \in [r]$. Also we fix $n_M = \sum_{i=1}^{k} |A_i|$ and write $r_M = r$ to emphasise that r is a function of M. Then for any monophonic position set M of G we have $|M| = n_M + r_M$. Now we are ready for our main result of this section.

Theorem 31

Let G be a connected graph of order at least 2 and let $\mathcal{M}$ be the collection of all monophonic position sets of G. Then

$$\mathrm{mp}(G \circ H) = \max_{M \in \mathcal{M}} \{n_M \cdot \omega(H) + r_M \cdot \mathrm{mp}(H)\}.$$

- Let S be a monophonic position set of $G \circ H$ and fix $M = \pi_G(S)$
- We first show that if $u \in M$ lies in a component A_i of $G[M]$ of order greater than one, then $\pi_H({}^uH \cap S)$ is a clique.
- Hence $|S| \leq \max_{M \in \mathcal{M}}\{n_M \cdot \omega(H) + r_M \cdot \mathrm{mp}(H)\}$

  For the converse, we show that any subset S of $V(G \circ H)$ with the properties that:

- $M = \pi_G(S)$ is in monophonic position in G,
- $\pi_H({}^uH \cap S)$ is a clique of H if u belongs to a component of $G[M]$ of order greater than one, and
- $\pi_H({}^uH \cap S)$ is a monophonic position set of H if u is an isolated vertex of $G[M]$,

  Then S is a monophonic position set of $G \circ H$

Any monophonic position set of a triangle-free graph G is either an independent set or consists of a pair of adjacent vertices, so if the order of the graph is $n \geq 3$ we have $\mathrm{mp}(G) = \mathrm{mp}_i(G)$. As $\omega(H) \leq \mathrm{mp}(H)$ this provides an exact value for $\mathrm{mp}(G \circ H)$ when G is triangle-free.

Corollary 32

If G be a connected triangle-free graph with order $n \geq 3$ and H is a connected graph, then $\mathrm{mp}(G \circ H) = \mathrm{mp}(G) \cdot \mathrm{mp}(H)$.

Corollary 33

If H is a connected graph and $n \geq 2$, then

$$\mathrm{mp}(K_n \circ H) = \max\{n \cdot \omega(H), \mathrm{mp}(H)\}.$$

References I

- Anand, B.S., Chandran S.V., U., Changat, M., Klavžar, S. & Thomas E.J., *Characterization of general position sets and its applications to cographs and bipartite graphs.* Appl. Math. Comput. 359 (2019), 84-89.
- Chandran S.V., U. & Parthasarathy, G.J., *The geodesic irredundant sets in graphs.* Int. J. Math. Comb. 4 (2016), 135-143.
- Manuel, P. & Klavžar, S., *A general position problem in graph theory.* Bull. Aust. Math. Soc. 98 (2) (2018), 177-187.
- Manuel, P. & Klavžar, S. *The graph theory general position problem on some interconnection networks.* Fundam. Inform. 163 (4) (2018), 339-350.

References II

- Thomas, E.J. & Chandran S.V., U., *Characterization of classes of graphs with large general position number.* AKCE Int. J. Graphs Comb. (2020), 1-5.
- E.J. Thomas, U. Chandran S.V., J. Tuite, G. Di Stefano, On monophonic position sets in graphs, Discrete Appl. Math. 354 (2024) 72–82.
- U. Chandran S.V., S. Klavžar, J. Tuite, The general position problem: a survey, arXiv:2501.19385 (2025).
- U. Chandran S.V., S. Klavžar, P.K. Neethu, J. Tuite, On monophonic position sets of Cartesian and lexicographic products of graphs, arXiv:2412.09837v2 [math.CO] (2025).

Thank you!