# Saturated Partial Embeddings of Planar Graphs

Alexander Clifton

Discrete Mathematics Group, Institute for Basic Science

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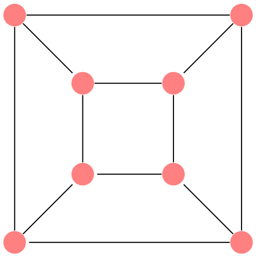
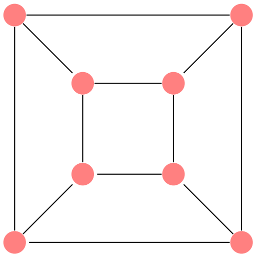
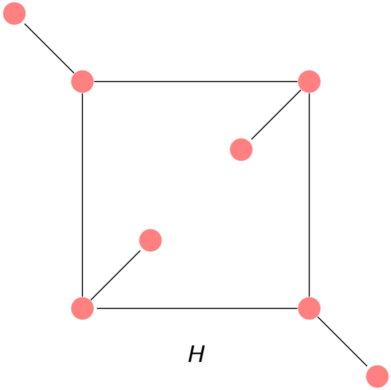
Joint work with Nika Salia



$G$



*G*



*H*

A *plane graph* is a planar graph **with a given embedding**.

For a planar graph $G$ and a plane subgraph $H \subseteq G$, we say that $H$ is a *plane-saturated subgraph* of $G$ if adding any edge to $H$ either:

- Forces a crossing, or
- Forces the resulting graph to no longer be a subgraph of $G$

The *plane saturation ratio*, $\text{psr}(G)$, is the minimum value of $\frac{|E(H)|}{|E(G)|}$ among all plane-saturated subgraphs $H$ of $G$.

**Example**

For a tree $T$, $\text{psr}(T) = 1$.

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$\text{psr}(Q^3) = \frac{8}{12}$.

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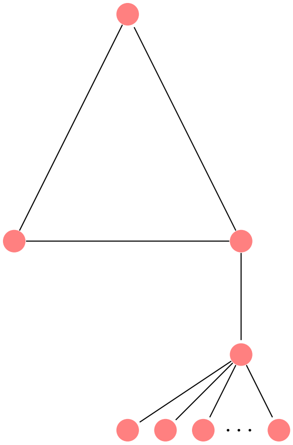
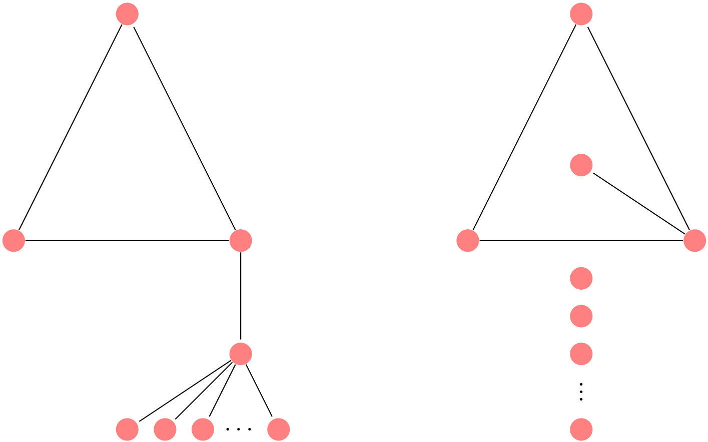
psr$(Q^3) = \frac{8}{12}$.

Our goal is to minimize psr$(G)$.

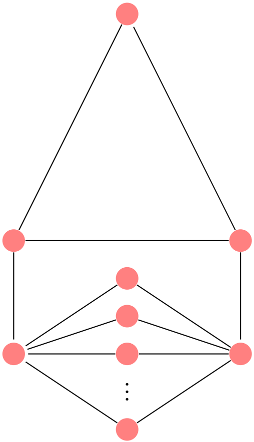


$G_n$



$G_n$ $H_n$

As $n \to \infty$, $\text{psr}(G_n) \to 0$.



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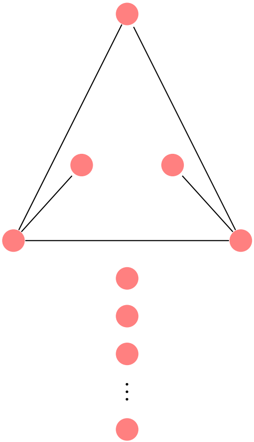

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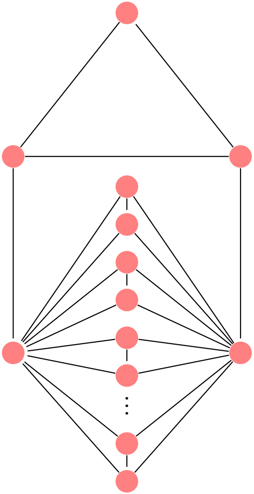
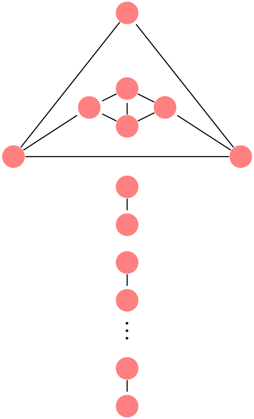
Let $\mathcal{G}_{k_1,k_2}$ denote the class of planar graphs where at most $k_1$ degree 1 vertices have the same neighborhood and at most $k_2$ degree 2 vertices have the same neighborhood. What is the minimum value of psr($G$) for $G \in \mathcal{G}_{k_1,k_2}$?



$G_n$



$H_n$

As $n \to \infty$, $\frac{|E(H_n)|}{|E(G_n)|} \to 1/5$.

## General Result

**Theorem (C.–Salia, 2023+)**

*For a positive integer $k_1$ and a nonnegative integer $k_2$ with $(k_1, k_2) \neq (1,0),(2,0)$, every planar graph $G \in \mathcal{G}_{k_1,k_2}$ satisfies*

$$\mathrm{psr}(G) > \frac{1}{9 + k_1 + 6k_2}.$$

*Furthermore, for every positive $\epsilon$, there exists a graph $G_\varepsilon \in \mathcal{G}_{k_1,k_2}$ such that*

$$\mathrm{psr}(G_\epsilon) < \frac{1}{9 + k_1 + 6k_2} + \epsilon.$$

# Main Result

**Theorem (C.–Salia, 2023+)**

*Every twin-free planar graph $G$ satisfies*

$$\mathrm{psr}(G) > 1/16.$$

*Furthermore, there exist twin-free planar graphs such that* $\mathrm{psr}(G)$ *is arbitrarily close to* $1/16$.

# Notation

Let $I$ be the set of isolated vertices of $H$. We collectively refer to the connected components of $H$ with more than one vertex as the *skeleton* and let $S$ denote the set of skeleton vertices.

There is at least one embedding $\phi : V(H) \to V(G)$ such that whenever $u, v \in V(H)$ are adjacent, we have that $\phi(u), \phi(v) \in V(G)$ are adjacent.

At various points in the proof, we may choose to consider a different embedding with this property.

# One isolated vertex per face

**Lemma**

*If a plane-saturated subgraph $H$ of $G$ contains no pair of isolated vertices within the same face, then* $\frac{|E(H)|}{|E(G)|} \geq 1/6$.

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**Proof.**

Let $H'$ be the induced subgraph of $H$ on the skeleton vertices and let $V(H'), E(H')$, and $F(H')$ be the sets of vertices, edges, and faces of $H'$ respectively.

Since $H'$ is planar, we have that $|V(H')| + |F(H')| = |E(H')| + k + 1$ where $k$ is the number of connected components of $H'$. Each component contains an edge, so $k \leq |E(H')|$. Thus, $|V(H')| + |F(H')| \leq 2|E(H')| + 1$.

□

# One isolated vertex per face

**Lemma**

*If a plane-saturated subgraph H of G contains no pair of isolated vertices within the same face, then* $\frac{|E(H)|}{|E(G)|} \geq 1/6$.

**Cont.**

Because $|I| \leq |F(H')|$, we have that

$$|V(G)| = |V(H)| \leq |V(H')| + |F(H')| \leq 2|E(H')| + 1 = 2|E(H)| + 1.$$

Since $G$ is planar, we have $|E(G)| \leq 3(2|E(H)| + 1) - 3 = 6|E(H)|$. Thus, $\frac{|E(H)|}{|E(G)|} \geq 1/6 > 1/16$. □

# Tabulating $|E(G)|$

There are three possible types of edges in $G$: those between two vertices of $\phi(S)$, those between two vertices of $\phi(I)$, and those between one of each.

**Lemma**

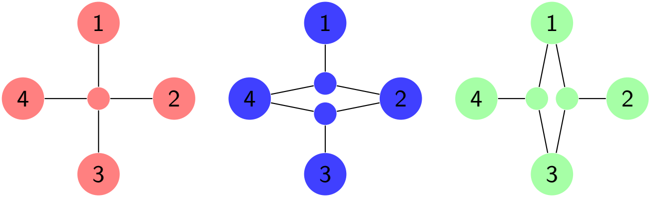
*If there are multiple vertices of $I$ within some face of $H$, then $\phi(I)$ is an independent set in $G$.*

**Proof.**

If such an edge existed in $G$, you could add a corresponding edge inside that face of $H$ without introducing a crossing and that would mean the original $H$ was not a plane-saturated subgraph of $G$, a contradiction. □

# Bounding the degrees in $\phi(I)$

It suffices to assume that every vertex in $\phi(I)$ has degree at most 3.



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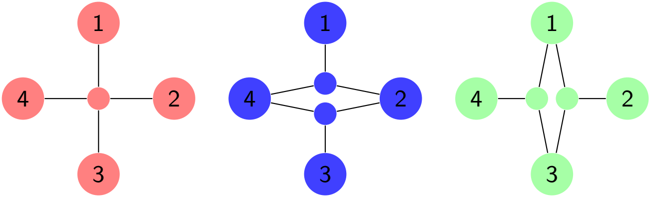


Splitting a high-degree vertex forms $G'$ with more edges than $G$. We can make a plane-saturated subgraph $H' \subseteq G'$ simply by adding an isolated vertex inside a face of $H$ that already has multiple isolated vertices within it. $H$ and $H'$ have the same skeleton so $\frac{|E(H')|}{|E(G')|} < \frac{|E(H)|}{|E(G)|}$.

# Bounding $|E(G)|$

For $i = 1, 2, 3$, let $J_i$ be the set of vertices in $\phi(I)$ with degree $i$.

$$|E(G)| = |E(G[\phi(S)])| + |J_1| + 2|J_2| + 3|J_3|.$$

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$G[\phi(S)]$ is planar so $|E(G[\phi(S)])| < 3|\phi(S)| = 3|S|$.

# Bounding $|J_1|$

$G$ has no degree 1 twins so no two vertices of $J_1$ have the same neighborhood. That means no vertex in $\phi(S)$ has multiple neighbors in $J_1$.

$$|J_1| \leq |S|.$$

# Bounding $|J_2|$

We can construct an auxiliary graph on the vertex set $\phi(S)$ as follows:

If $\{u, v\}$ is the neighborhood of some vertex in $J_2$, then $uv$ is an edge in the auxiliary graph.

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No two vertices of $J_2$ have the same neighborhood so this is a simple graph. Furthermore, $G$ is planar, so the auxiliary graph is too.

$|J_2|$ is the number of edges in the auxiliary graph, so

$$|J_2| \leq 3|S|.$$

# Bounding $|J_3|$

Consider the subgraph of $G$ whose edge set is all edges incident to $J_3$ (and whose vertex set consists of the vertices incident to those edges.)

$J_3 \subseteq \phi(I)$ is an independent set, so this is a bipartite graph. Thus,

$$3|J_3| \leq 2\left(|S| + |J_3|\right),$$

so $|J_3| \leq 2|S|$.

# Putting it altogether

$$|E(G)| = |E(G[\phi(S)])| + |J_1| + 2|J_2| + 3|J_3| < 16|S|.$$

If every component of the skeleton has a cycle, then $|E(H)| \geq |S|$, giving us the desired bound of $\frac{|E(H)|}{|E(G)|} > 1/16$, but if some skeleton components are trees, we are not done yet.

Worst possible case comes from components with two vertices, so lower bound we have shown so far is $\frac{1}{32}$.

# Dealing with trees in the skeleton

For a tree with $t$ vertices, the contribution to $|E(H)|$ is $t - 1$, but the contribution to our upper bound on $|E(G)|$ is $16t$, which is greater than $16(t - 1)$.

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**Lemma**

*Let $C$ be a skeleton component of $H$ isomorphic to a tree, $u$ a leaf of $C$, and $v$ the sole neighbor of $u$ in $H$. It is impossible for $\phi(u), \phi(v)$ to have distinct neighbors in $\phi(I)$.*

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Our bounds on $|J_1|, |J_2|, |J_3|$ assumed that every vertex in $\phi(S)$ had neighbors in all three of $J_1, J_2, J_3$. However, for every pair of vertices considered in the Lemma, either one vertex has no neighbors in $\phi(I)$ or both have just one neighbor in $\phi(I)$, which achieves even more savings.

# Dealing with trees in the skeleton

For every vertex $v \in S$ where $\phi(S)$ has no neighbors in $\phi(I)$, we get savings of 13 in how we calculated the upper bound on $|E(G)|$.

We need savings of 16 for a component isomorphic to a tree, so it is enough to have two such vertices. We have leaves $u$ and $u'$ whose sole neighbors $v$ and $v'$ are distinct, as long as the tree is not a star.

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However, if component is isomorphic to a star, it is possible to still have a contribution of $16t - 13$ to the upper bound on $|E(G)|$ if the center of the star has no neighbors in $\phi(I)$, but every leaf does have neighbors in $\phi(I)$.

# Dealing with stars in the skeleton

To get the necessary savings for components isomorphic to such stars, we amend our bound on $|J_3|$. We obtained this by considering the bipartite subgraph of $G$ whose edges were precisely the edges incident to $J_3$.

For a star where the center has no neighbors in $\phi(I)$, we add the edges of this star to that subgraph. Indeed, it is still bipartite!

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Repeating the previous argument results in a contribution of just $13t - 4$ from such stars to the upper bound on $|E(G)|$.

$$\frac{t-1}{13t-4} \geq 1/16$$

when $t \geq 4$, and it is easy to deal with $t = 3$.

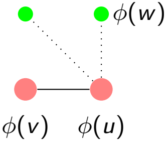
# Dealing with matching edges

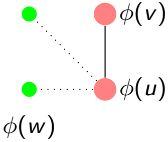
For a skeleton component on two vertices, $u$ and $v$, either $\phi(u), \phi(v)$ have a single shared neighbor in $\phi(I)$ or one has no neighbors in $\phi(I)$.

A matching edge contributes 1 to $|E(H)|$, but the second case contributes $3(2) + 1 + 6 + 6 = 19$ to our bound on $|E(G)|$. This is enough to establish that $\frac{|E(H)|}{|E(G)|} \geq 1/19$.

# Dealing with matching edges

It is possible to change the embedding of $H$ into $G$. If $\phi(u)$ has a neighbor $\phi(w)$ in $\phi(I)$ but $\phi(v)$ has no neighbors in $\phi(I)$, we can swap where $w$ and $v$ are embedded.




Embedding $v$ to as low a degree a vertex as possible can provide some savings in the $|E(G[\phi(S)])|$ and $|J_1|$ term.

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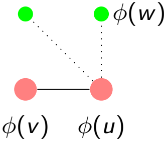
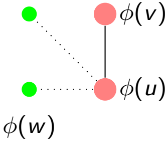



Embedding $v$ to as low a degree a vertex as possible can provide some savings in the $|E(G[\phi(S)])|$ and $|J_1|$ term. However, this is still only enough for $\frac{1}{17}$.

# Making it work

We revise our tabulations for $|E(G[\phi(S)])|$ and $2|J_2|$.

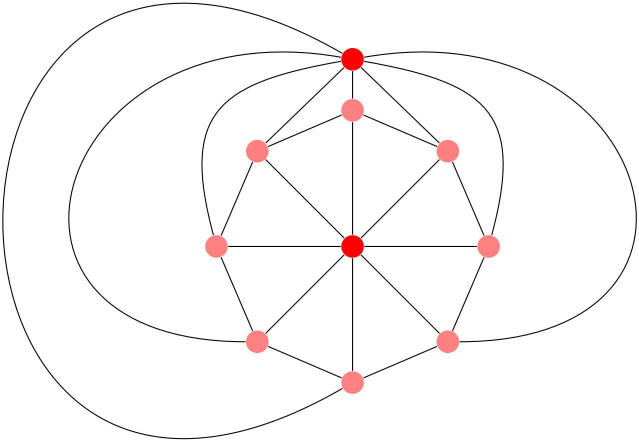
Let $R_2$ be the set of skeleton vertices $v$ where we know $\phi(v)$ has degree 2. We instead find bounds on $|E(G[\phi(S \setminus R_2)])|$ and $2|J_2 \cup R_2|$. However, this is only enough for $1/18$.

# Making it work

The lower bounds of $1/17$ and $1/18$ we found were both worst case scenarios and can be improved when certain types of matching edges are more prevalent than others.

This $1/18$ can be improved to $1/16$ when $|R_2|$ is sufficiently large relative to the number of matching edges where the image of one vertex has more than one neighbor in $\phi(I)$ but none of degree less than 3. When $|R_2|$ is too small to improve this bound, it is always small enough to improve the earlier $1/17$ bound to $1/16$.

# Upper Bound Construction



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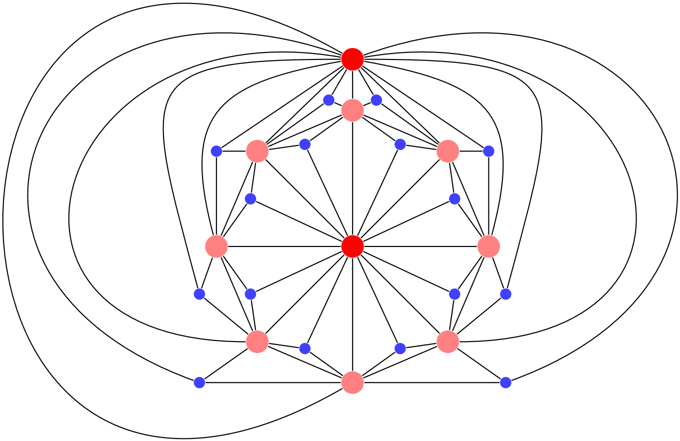

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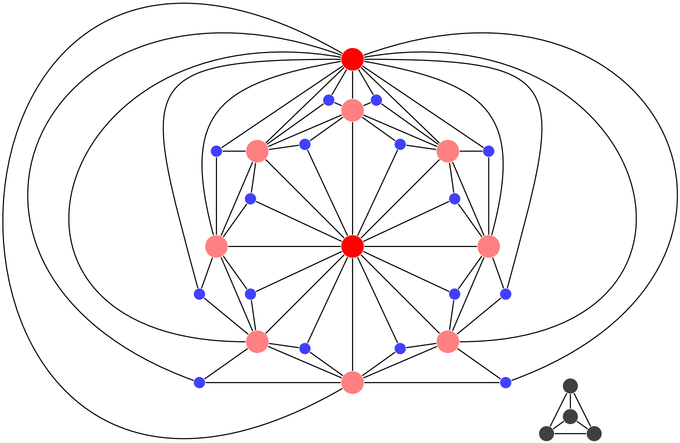

# Open Problems

- What about the remaining $(k_1, k_2)$ pairs?
- What is the minimum value of $\frac{|E(H)|}{|E(G)|}$ for a plane-saturated subgraph $H$ of $G$ with no isolated vertices?
- What happens when we extend this notion of saturated drawings to graphs drawn on other surfaces or to 1-planar graphs?

Thank you!

Any Questions?