Spectra and eigenspaces from regular partitions of Cayley (di)graphs of permutation groups

Cristina Dalfó
Universitat de Lleida, Igualada, Catalonia

Miquel Àngel Fiol
Universitat Politècnica de Catalunya, Barcelona, Catalonia

Open University Discrete Mathematics Seminar

January, 20, 2021

The first author has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 734922.

Outline

Outline

1. Introduction

2. Regular partitions of vertices from number partitions

3. The spectra of quotient digraphs

Goals

- To prove that **every partition of the number n** gives rise to a **regular (or equitable) partition of a Cayley (di)graph of permutation groups on n letters**.

Goals

- To prove that **every partition of the number n** gives rise to a **regular (or equitable) partition of a Cayley (di)graph of permutation groups on n letters**.
  Example:
  - 4=1+1+1+1,
  - 4=1+1+2,
  - 4=1+3,
  - 4=2+2,
  - 4=4.

Goals

- To prove that **every partition of the number n** gives rise to a **regular (or equitable) partition of a Cayley (di)graph of permutation groups on n letters**.
  Example:
  - 4=1+1+1+1,
  - 4=1+1+2,
  - 4=1+3,
  - 4=2+2,
  - 4=4.
- To obtain the **complete spectra and the eigenspaces of the quotient (di)graphs of regular partitions**.

Goals

- To prove that **every partition of the number n** gives rise to a **regular (or equitable) partition of a Cayley (di)graph of permutation groups on n letters**.
  Example:
  - 4=1+1+1+1,
  - 4=1+1+2,
  - 4=1+3,
  - 4=2+2,
  - 4=4.
- To obtain the **complete spectra and the eigenspaces of the quotient (di)graphs of regular partitions**.
- As an example, to apply this method to the **pancake graphs $P(n)$**.

Notation

- $\Gamma = (V, E)$: **(Di)graph** (graph, digraph, or mixed graph). It can have loops and multiple arcs.

Notation

- $\Gamma = (V, E)$: **(Di)graph** (graph, digraph, or mixed graph). It can have loops and multiple arcs.
- We consider **Cayley (di)graphs Cay**(G, S), where G is a subgroup of the symmetric group $S_n = \mathrm{Sym}(n)$, and S is the generating set given by some permutations $\pi_1, \pi_2, \ldots, \pi_k$.

Notation

- $\Gamma = (V, E)$: **(Di)graph** (graph, digraph, or mixed graph). It can have loops and multiple arcs.
- We consider **Cayley (di)graphs Cay**(G, S), where G is a subgroup of the symmetric group $S_n = \mathrm{Sym}(n)$, and S is the generating set given by some permutations $\pi_1, \pi_2, \ldots, \pi_k$.
- **Spectrum** of the adjacency matrix A of Γ:

$$\operatorname{sp}\Gamma = \{[\lambda_0]^{m_0}, [\lambda_1]^{m_1}, \ldots, [\lambda_d]^{m_d}\},$$

where λ_i and m_i are the different eigenvalues and their multiplicities.

Regular partitions and their spectra

- A partition $\pi = (V_1, \ldots, V_m)$ of its vertex set V is called **equitable** (or **regular**) whenever, for any $i, j = 1, \ldots, m$, the intersection numbers $b_{ij}(u) = |\Gamma(u) \cap V_j|$, where $u \in V_i$, do not depend on the vertex u but only on the subsets (called classes or cells) V_i and V_j. In this case, $b_{ij}(u) = b_{ij}$.

Regular partitions and their spectra

- A partition $\pi = (V_1, \ldots, V_m)$ of its vertex set V is called **equitable** (or **regular**) whenever, for any $i, j = 1, \ldots, m$, the intersection numbers $b_{ij}(u) = |\Gamma(u) \cap V_j|$, where $u \in V_i$, do not depend on the vertex u but only on the subsets (called classes or cells) V_i and V_j. In this case, $b_{ij}(u) = b_{ij}$.
- The $m \times m$ matrix $B = (b_{ij})$ is the **quotient matrix** of A with respect to π.

Regular partitions and their spectra

- A partition $\pi = (V_1, \ldots, V_m)$ of its vertex set V is called **equitable** (or **regular**) whenever, for any $i, j = 1, \ldots, m$, the intersection numbers $b_{ij}(u) = |\Gamma(u) \cap V_j|$, where $u \in V_i$, do not depend on the vertex u but only on the subsets (called classes or cells) V_i and V_j. In this case, $b_{ij}(u) = b_{ij}$.
- The $m \times m$ matrix $B = (b_{ij})$ is the **quotient matrix** of A with respect to π.
- The **quotient (weighted) digraph** $\pi(\Gamma)$ (associated to the partition π), has vertices representing the cells, and there is an edge with weight b_{ij} from vertex V_i to vertex V_j if and only if $b_{ij} \neq 0$. If $b_{ii} > 0$, for some $i = 1, \ldots, m$, the quotient graph $\pi(\Gamma)$ has loops.

The characteristic matrix

- The **characteristic matrix** of a partition π is the $n \times m$ matrix $\boldsymbol{S} = (s_{ui})$ whose i-th column is the characteristic vector of V_i:

$$s_{ui} = \begin{cases} 1 & \text{if } u \in V_i, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic matrix

- The **characteristic matrix** of a partition π is the $n \times m$ matrix $\boldsymbol{S} = (s_{ui})$ whose i-th column is the characteristic vector of V_i:

$$s_{ui} = \begin{cases} 1 & \text{if } u \in V_i, \\ 0 & \text{otherwise.} \end{cases}$$

- Lemma. [Godsil, 1993]
  Let $\Gamma = (V, E)$ be a digraph with adjacency matrix A, and vertex partition π with characteristic matrix S.
  - (i) The partition π is **regular** if and only if $SB = AS$, where B is the **quotient** matrix of A with respect to π.
  - (ii) If π is regular and x is an eigenvector of B, then Sx is an eigenvector of A. Consequently,

$$\text{sp}\, B \subseteq \text{sp}\, A.$$

Lifted digraphs and their spectra

- Given a digraph Γ, and a finite group G with generating set S, a **voltage assignment** α is a mapping $\alpha : E \to S$, that is, a labeling of the arcs with the elements of S.

Lifted digraphs and their spectra

- Given a digraph Γ, and a finite group G with generating set S, a **voltage assignment** α is a mapping $\alpha : E \to S$, that is, a labeling of the arcs with the elements of S.
- $\Gamma : u \longrightarrow v$.

Lifted digraphs and their spectra

- Given a digraph Γ, and a finite group G with generating set S, a **voltage assignment** α is a mapping $\alpha : E \to S$, that is, a labeling of the arcs with the elements of S.

- $\Gamma : u \longrightarrow v$.

- **Lifted digraph** Γ^α: Digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times S$, where there is an arc from vertex (u, g) to vertex $(v, g\alpha(uv))$ if and only if $uv \in E$:

$$(u, g) \longrightarrow (v, g\alpha(uv)).$$

Lifted digraphs and their spectra

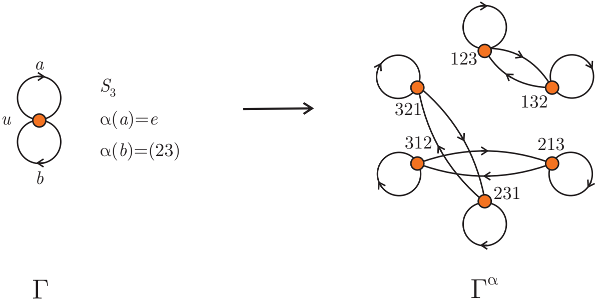
- Given a digraph Γ, and a finite group G with generating set S, a **voltage assignment** α is a mapping $\alpha : E \to S$, that is, a labeling of the arcs with the elements of S.
- $\Gamma : u \longrightarrow v$.
- **Lifted digraph** Γ^α: Digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times S$, where there is an arc from vertex (u, g) to vertex $(v, g\alpha(uv))$ if and only if $uv \in E$:

$$(u, g) \longrightarrow (v, g\alpha(uv)).$$

- In particular, the **Cayley digraph** $\mathrm{Cay}(\Gamma, S)$ with $S = \{g_1, \ldots, g_r\}$ can be seen as the lifted digraph Γ^α, where $\Gamma = K_1^r$ (a singleton with $V = \{u\}$ and $E = \{e_1, \ldots, e_r\}$ are r loops) and voltage assignment

$$\begin{aligned} \alpha : E &\longrightarrow S \\ e_i &\longrightarrow \alpha(e_i) = g_i \end{aligned}$$

An example of a lifted digraph



The base matrix

To the pair (Γ, α), we assign the $k \times k$ **base matrix** B, a square matrix whose rows and columns are indexed by elements of the vertex set of Γ, and whose uv-th element B_{uv} is determined as follows:

- If $a_1, \ldots, a_j$ is the set of all the arcs of Γ emanating from u and terminating at v (not excluding the case $u = v$), then

$$B_{uv} = \alpha(a_1) + \cdots + \alpha(a_j),$$

  the sum being an element of the complex group algebra $\mathbb{C}(G)$.
- Otherwise, $B_{uv} = 0$.

The expanded base matrix

Given a **unitary irreducible representation** of the group G, $\rho \in \mathrm{Irep}(G)$, of dimension d_ρ, let $\rho(B)$ be the $d_\rho k \times d_\rho k$ matrix obtained from B by replacing every entry $B_{u,v} \in \mathbb{C}(G)$ by the $d_\rho \times d_\rho$ matrix

$$\rho(B_{u,v}) = \begin{cases} \rho(\alpha(a_1)) + \cdots + \rho(\alpha(a_j)) & \text{if } B_{u,v} \neq 0, \\ O & \text{otherwise,} \end{cases}$$

where O is the all-zero $d_\rho \times d_\rho$ matrix.

Conjugacy classes

- The elements of any group may be partitioned into **conjugacy classes**. The members of the same conjugacy class share properties.

Conjugacy classes

- The elements of any group may be partitioned into **conjugacy classes**. The members of the same conjugacy class share properties.
- **Example:** The symmetric group S_3, consisting of 6 permutations of 3 elements, has 3 conjugacy classes:
  - No change: $abc \to abc$.
  - Transposing 2: $abc \to acb,\ abc \to bac,\ abc \to cba$.
  - A cyclic permutation of all 3: $abc \to bca,\ abc \to cab$.

Conjugacy classes

- The elements of any group may be partitioned into **conjugacy classes**. The members of the same conjugacy class share properties.
- **Example:** The symmetric group S_3, consisting of 6 permutations of 3 elements, has 3 conjugacy classes:
  - No change: $abc \to abc$.
  - Transposing 2: $abc \to acb,\ abc \to bac,\ abc \to cba$.
  - A cyclic permutation of all 3: $abc \to bca,\ abc \to cab$.
- The number of conjugacy classes in the symmetric group S_n is equal to the number of integer partitions of n.

The spectrum of a (regular) lifted (di)graph

Theorem (D., Fiol, Širáň, 2019)

Let $\Gamma = (V, E)$ *be a base digraph on* k *vertices, with a voltage assignment* α *in a group* G*, with* $|G| = n$*.*

The spectrum of a (regular) lifted (di)graph

Theorem (D., Fiol, Širáň, 2019)

Let $\Gamma = (V, E)$ *be a base digraph on* k *vertices, with a voltage assignment* α *in a group* G*, with* $|G| = n$.

Assume that G *has* ν *conjugacy classes with dimensions* $d_1, \ldots, d_\nu$ *(so,* $\sum_{i=1}^{\nu} d_i^2 = n$*).*

The spectrum of a (regular) lifted (di)graph

Theorem (D., Fiol, Širáň, 2019)

Let $\Gamma = (V, E)$ *be a base digraph on* k *vertices, with a voltage assignment* α *in a group* G*, with* $|G| = n$*.*

Assume that G *has* ν *conjugacy classes with dimensions* $d_1, \ldots, d_\nu$ *(so,* $\sum_{i=1}^{\nu} d_i^2 = n$*).*

Let $\rho_i \in \operatorname{Irep} G$ *be irreducible representations of the group* G*, for* $i = 1, \ldots, \nu$*. Let* $\rho_i(B)$ *be the complex matrix with entries given by*

$$\rho(B_{u,v}) = \begin{cases} \rho(\alpha(a_1)) + \cdots + \rho(\alpha(a_j)) & \textit{if } B_{u,v} \neq 0, \\ O & \textit{otherwise}, \end{cases}$$

The spectrum of a (regular) lifted (di)graph

Theorem (D., Fiol, Širáň, 2019)

Let $\Gamma = (V, E)$ be a base digraph on k vertices, with a voltage assignment α in a group G, with $|G| = n$.

Assume that G has ν conjugacy classes with dimensions $d_1, \ldots, d_\nu$ (so, $\sum_{i=1}^{\nu} d_i^2 = n$).

Let $\rho_i \in \mathrm{Irep}\, G$ be irreducible representations of the group G, for $i = 1, \ldots, \nu$. Let $\rho_i(B)$ be the complex matrix with entries given by

$$\rho(B_{u,v}) = \begin{cases} \rho(\alpha(a_1)) + \cdots + \rho(\alpha(a_j)) & \text{if } B_{u,v} \neq 0, \\ O & \text{otherwise,} \end{cases}$$

Then, the kn eigenvalues of the lift Γ^α are the kd_i eigenvalues of $\rho_i(B)$, for every $i \in [\nu]$, each repeated d_i times.

A generalization for (not necessarily regular) lifts.

- Let $\Gamma = (V, E)$ be a digraph, G a group, and H a subgroup of G of index n. Let G/H denote the set of right cosets of H in G. Let $\beta : E \to G$ be the so-called **relative voltage assignment**.

A generalization for (not necessarily regular) lifts.

- Let $\Gamma = (V, E)$ be a digraph, G a group, and H a subgroup of G of index n. Let G/H denote the set of right cosets of H in G. Let $\beta : E \to G$ be the so-called **relative voltage assignment**.
- $\Gamma : \; u \;\longrightarrow\; v$.
  $\Gamma^{\alpha} : \; (u, g) \;\longrightarrow\; (v, g\alpha(uv))$.

A generalization for (not necessarily regular) lifts.

- Let $\Gamma = (V, E)$ be a digraph, G a group, and H a subgroup of G of index n. Let G/H denote the set of right cosets of H in G. Let $\beta : E \to G$ be the so-called **relative voltage assignment**.
- $\Gamma : \ u \ \longrightarrow \ v$.
  $\Gamma^{\alpha} : \ (u, g) \ \longrightarrow \ (v, g\alpha(uv))$.
- The **relative lift (with respect to H)** Γ^{β} has vertex set $V^{\beta} = V \times G/H$ and arc set $E^{\beta} = E \times G/H$, and, if a is an arc from a vertex u to a vertex v in Γ, then for every right coset $J \in G/H$ there is an arc (a, J):

$$(u, J) \ \longrightarrow \ (v, J\beta(a)).$$

A generalization for (not necessarily regular) lifts.

- Let $\Gamma = (V, E)$ be a digraph, G a group, and H a subgroup of G of index n. Let G/H denote the set of right cosets of H in G. Let $\beta : E \to G$ be the so-called **relative voltage assignment**.

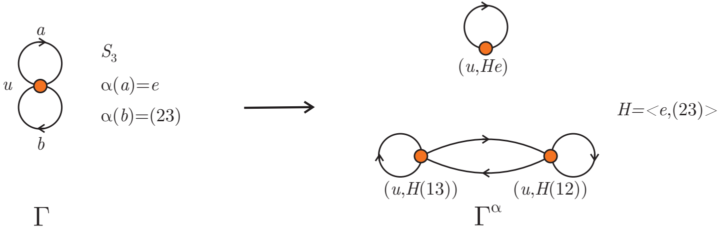
- $\Gamma: \ u \ \longrightarrow \ v.$
  $\Gamma^{\alpha}: \ (u, g) \ \longrightarrow \ (v, g\alpha(uv)).$

- The **relative lift (with respect to H)** Γ^{β} has vertex set $V^{\beta} = V \times G/H$ and arc set $E^{\beta} = E \times G/H$, and, if a is an arc from a vertex u to a vertex v in Γ, then for every right coset $J \in G/H$ there is an arc (a, J):

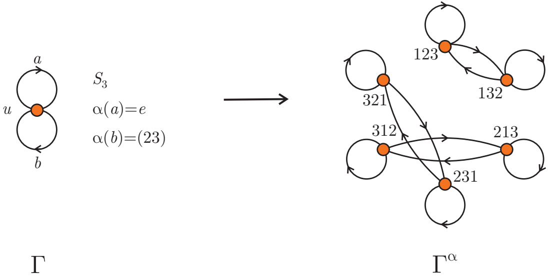
$$(u, J) \ \longrightarrow \ (v, J\beta(a)).$$

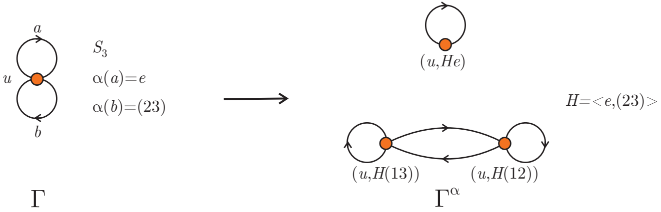
- A relative voltage assignment β in a group G with subgroup H is equivalent to an 'ordinary' voltage assignment if and only if H is a **normal subgroup** of G.

An example of a relative lifted graph



Examples of lifted (di)graph/relative lifted (di)graph





A general result

Theorem (D., Fiol, Pavlíková, Širáň, 2019)

Let Γ be a base digraph of order k and let α be a voltage assignment on Γ in a group G relative to a subgroup H of index n in G. Given an irreducible representation $\rho \in \mathrm{Irep}(G)$, let us consider the matrix $\rho(H) = \sum_{h\in H} \rho(h)$. Then,

$$\mathrm{sp}\,\Gamma^{\alpha} = \bigcup_{\rho\in \mathrm{Irep}(G)} \mathrm{rank}(\rho(H)) \cdot \mathrm{sp}(\rho(B)),$$

where the union must be understood for all $\rho \in \mathrm{Irep}(G)$ such that $\mathrm{rank}(\rho(H)) \neq 0$.

An example: The pancake graphs

The n-**dimensional pancake graph** $P(n)$, proposed by Dweighter (1975), is a graph with vertex set

$$V(P(n)) = \{x_1x_2 \ldots x_n | x_i \in [n], x_i \neq x_j \text{ for } i \neq j\}.$$

An example: The pancake graphs

The n-**dimensional pancake graph** $P(n)$, proposed by Dweighter (1975), is a graph with vertex set

$$V(P(n)) = \{x_1x_2\dots x_n | x_i \in [n], x_i \neq x_j \text{ for } i \neq j\}.$$

Its adjacencies are:

$$x_1x_2\dots x_n \sim \begin{cases} x_1\dots x_{n-2}x_nx_{n-1}, \\ x_1\dots x_{n-3}x_nx_{n-1}x_{n-2}, \\ x_1\dots x_{n-4}x_nx_{n-1}x_{n-2}x_{n-3}, \\ \vdots \\ x_nx_{n-1}\dots x_2x_1. \end{cases}$$

The first pancake graphs

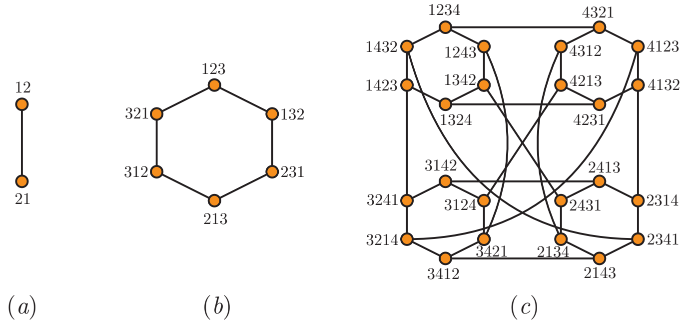


Figure: Pancakes graphs: (a) $P(2)$, (b) $P(3)$, and (c) $P(4)$.

The pancake graphs $P(n)$: Properties

- $P(n)$ are vertex-transitive.

The pancake graphs $P(n)$: Properties

- $P(n)$ are vertex-transitive.
- $P(n)$ are $(n-1)$-regular.

The pancake graphs $P(n)$: Properties

- $P(n)$ are vertex-transitive.
- $P(n)$ are $(n-1)$-regular.
- $P(n)$ have $n!$ vertices.

The pancake graphs $P(n)$: Properties

- $P(n)$ are vertex-transitive.
- $P(n)$ are $(n-1)$-regular.
- $P(n)$ have $n!$ vertices.
- $P(n)$ are **Cayley graphs** Cay(G, S), where G is the symmetric group S_n and the generating set S corresponds to the permutations of $x_1, x_2, \ldots, x_n$ given by the adjacencies of P_n.

The pancake graphs $P(n)$: Diameter

- The exact diameters $k = k(n)$ of $P(n)$ are only known for $n \leq 17$ (Cibulka 2011, Sloane 2007):

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
k	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19

The pancake graphs $P(n)$: Diameter

- The exact diameters $k = k(n)$ of $P(n)$ are only known for $n \leq 17$ (Cibulka 2011, Sloane 2007):

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
k	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19

- The best bounds were given by Gates and Papadimitriou (1979):

$$\frac{17}{16}n \leq k(n) \leq \frac{5n+5}{3}.$$

The pancake graphs $P(n)$: Diameter

- The exact diameters $k = k(n)$ of $P(n)$ are only known for $n \leq 17$ (Cibulka 2011, Sloane 2007):

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
k	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19

- The best bounds were given by Gates and Papadimitriou (1979):

$$\frac{17}{16}n \leq k(n) \leq \frac{5n+5}{3}.$$

- Heydari and Sudborough (1997) improved the lower bound to

$$k(n) \geq \frac{15}{14}n.$$

Outline

1. Introduction

2. Regular partitions of vertices from number partitions

3. The spectra of quotient digraphs

Regular partitions of vertices from number partitions

Given a permutation $\pi : [n] \to [n]$, we denote by $P(\pi) = (p_{ij})$ the $n \times n$ **permutation matrix** (called column representation) with entries

$$p_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Regular partitions of vertices from number partitions

Given a permutation $\pi : [n] \to [n]$, we denote by $P(\pi) = (p_{ij})$ the $n \times n$ **permutation matrix** (called column representation) with entries

$$p_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition

Let $\Gamma =$Cay(G, S) be a Cayley digraph, where G is a subgroup of the symmetric group S_n and its generating set S is given by the permutations $\{\pi_1, \pi_2, \ldots, \pi_k\}$. Then, Γ has a regular partition β with **quotient matrix**

$$B = \sum_{i=1}^{k} P(\pi_i).$$

Then, by Godsil's Lemma (ii),

$$\mathrm{sp}\, B \subset \mathrm{sp}\, A(\Gamma).$$

An example: The Pancake graph $P(4)$

- Let $P(4) \cong \mathrm{Cay}(S_4, S)$ with $S = \{(34), (24), (14)(23)\}$.

An example: The Pancake graph $P(4)$

- Let $P(4) \cong \mathrm{Cay}(S_4, S)$ with $S = \{(34), (24), (14)(23)\}$.
- The sum of the corresponding permutation matrices is

$$\begin{aligned} B &= P((34)) + P((24)) + P((14)(23)) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \end{aligned}$$

which is **the quotient matrix of a regular partition** of $P(4)$.

An example: The Pancake graph $P(4)$

- Let $P(4) \cong \mathrm{Cay}(S_4, S)$ with $S = \{(34), (24), (14)(23)\}$.
- The sum of the corresponding permutation matrices is

$$\begin{aligned} B &= P((34)) + P((24)) + P((14)(23)) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \end{aligned}$$

  which is **the quotient matrix of a regular partition** of $P(4)$.
- $\mathrm{sp}\, B = \{3, 2, 0, -1\} \subset \mathrm{sp}\, P(4)$.

An example: The Pancake graph $P(4)$

- Let $P(4) \cong \mathrm{Cay}(S_4, S)$ with $S = \{(34), (24), (14)(23)\}$.
- The sum of the corresponding permutation matrices is

$$
\begin{aligned}
B &= P((34)) + P((24)) + P((14)(23)) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},
\end{aligned}
$$

which is **the quotient matrix of a regular partition** of $P(4)$.

- $\mathrm{sp}\, B = \{3, 2, 0, -1\} \subset \mathrm{sp}\, P(4)$.
- $\mathrm{sp}\, P(4) = \left\{ [3]^1, [2]^5, \left[\frac{-1+\sqrt{17}}{2}\right]^3, [0]^5, \left[\frac{-1-\sqrt{17}}{2}\right]^3, [-1]^4, [-2]^3 \right\}$.

An example: The Pancake graph $P(4)$

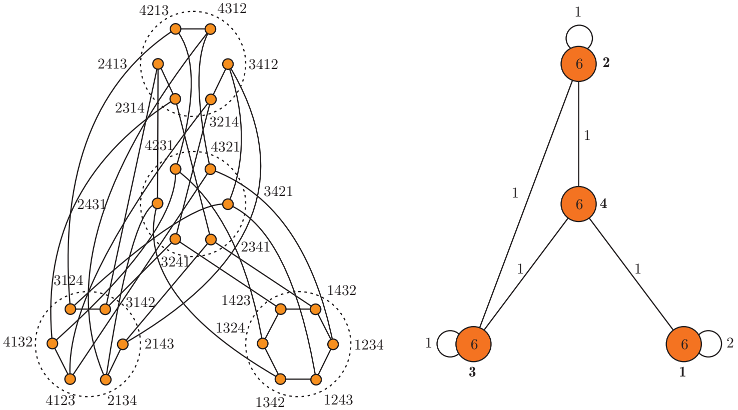


Figure: A regular partition of $P(4)$ and its quotient graph.

The case of $P(n)$ for any n

Proposition

(i) *The matrix* $B_n = \sum_{i=1}^{n} P(\pi_i)$ *of the pancake graph* $P(n)$ *is the sum* $B_n = D_n + T_n$*, where*

$$D_n = \begin{pmatrix} n-2 & & & & \\ & n-3 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \\ & & & & -1 \end{pmatrix}, \quad T_n = \begin{pmatrix} & & & & 1 \\ & 0 & & 1 & \\ & & \iddots & & \\ & 1 & & 1 & \\ 1 & & & & \end{pmatrix}.$$

The case of $P(n)$ for any n

Proposition

(i) *The matrix $B_n = \sum_{i=1}^{n} P(\pi_i)$ of the pancake graph $P(n)$ is the sum $B_n = D_n + T_n$, where*

$$D_n = \begin{pmatrix} n-2 & & & & \\ & n-3 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \\ & & & & -1 \end{pmatrix}, \quad T_n = \begin{pmatrix} & & & 1 \\ & 0 & 1 & \\ & & \iddots & \\ & 1 & 1 & \\ 1 & & & \end{pmatrix}.$$

Its **spectrum** *is*

$$\operatorname{sp} B_n = \{n-1, n-2, \ldots, 0, -1\} \setminus \{\lfloor (n/2) - 1 \rfloor\},$$

all the eigenvalues with multiplicity 1.

The case of $P(n)$ for any n

Proposition

(ii) *Their* **eigenvectors** *are, respectively,*

- *the all-1 vector* $(1, 1, \ldots, 1)^\top$,
- $\left(0, \overset{(r-1)}{\ldots}, 0, n-2r, -1, \overset{(n-2r)}{\ldots}, -1, 0, \overset{(r)}{\ldots}, 0\right)^\top$
  for $\begin{cases} r = 1, \ldots, \lfloor n/2 \rfloor & (n \text{ odd}), \\ r = 1, \ldots, \lfloor n/2 \rfloor - 1 & (n \text{ even}), \end{cases}$
- $\left(0, \overset{(r-1)}{\ldots}, 0, -1, \overset{(n-2r+1)}{\ldots}, -1, n-2r+1, 0, \overset{(r-1)}{\ldots}, 0\right)^\top$,
  for $r = 1, \ldots, \lfloor n/2 \rfloor$.

An example: The Pancake graph $P(5)$

$$B_5 = P((45)) + P((35)) + P((25)(34)) + P((15)(24))$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= D_5 + T_5.$$

An example: The Pancake graph $P(5)$

$$\operatorname{sp} B_5 = \{4, 3, 2, 0, -1\},$$

and the corresponding matrix of (column) eigenvectors is

$$\begin{pmatrix} 1 & 3 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

A general result

Let $\mathcal{PR} = PR_n^{n_1,\dots,n_r}$ denote the set of **permutations with repetitions** of r symbols $a, b, \dots$, where a is repeated n_1 times, b is repeated n_2 times, etc., with $n_1 + n_2 + \cdots + n_r = n$. Then, $|\mathcal{PR}| = \frac{n!}{n_1! \cdots n_r!}$.

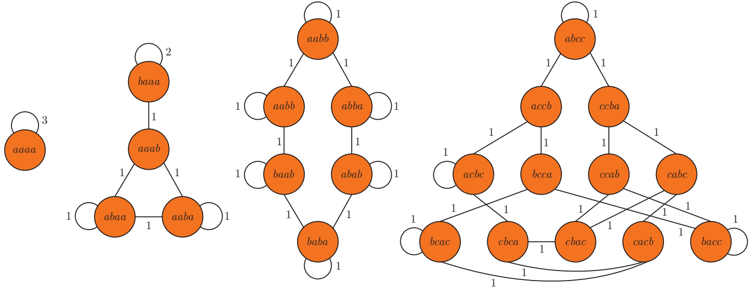
A general result

Let $\mathcal{PR} = PR_n^{n_1,\ldots,n_r}$ denote the set of **permutations with repetitions** of r symbols $a, b, \ldots$, where a is repeated n_1 times, b is repeated n_2 times, etc., with $n_1 + n_2 + \cdots + n_r = n$. Then, $|\mathcal{PR}| = \frac{n!}{n_1! \cdots n_r!}$.

Theorem

Let $\Gamma =$Cay(G, S) be a Cayley digraph, where G is a subgroup of the symmetric group S_n and its generating set S is given by the permutations $\{\pi_1, \pi_2, \ldots, \pi_k\}$. **For any partition $n_1 + n_2 + \cdots + n_r = n$, there is a regular partition of Γ** *with quotient matrix B indexed by elements of $\mathcal{PR}$, and for every $\sigma, \tau \in \mathcal{PR}$ the entry $(B)_{\sigma\tau}$ is the number (possibly zero) of permutations in S that, acting on the symbol positions, map σ into τ.*

An example: $P(4)$



4=4 4=3+1 4=2+2 4=2+1+1

Figure: The regular partitions of $P(4)$ corresponding to the number partitions of 4, with number of classes $PR_4^4 = 1$, $PR_4^{3,1} = 4$, $PR_4^{2,2} = 6$, and $PR_4^{2,1,1} = 12$. The case $PR_4^{1,1,1,1}$, corresponding to $P(4)$, is not drawn here.

Outline

1. Introduction
2. Regular partitions of vertices from number partitions
3. The spectra of quotient digraphs

The spectra of quotient digraphs

- **Goal:** We want to find the **whole spectrum** of a Cayley digraph (on a permutation group) and their quotient digraphs associated to the corresponding partitions.

The spectra of quotient digraphs

- **Goal:** We want to find the **whole spectrum** of a Cayley digraph (on a permutation group) and their quotient digraphs associated to the corresponding partitions.
- Given g in the group G and x in the set X with $g \cdot x = x$, we say x is a **fixed point** of g, and g fixes x.
  For every x in X, the **stabilizer subgroup** of G with respect to x is the set of all elements in G that fix x:

$$G_x = \{g \in G | g \cdot x = x\}.$$

The spectra of quotient digraphs

Lemma

Let $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph, where* G *is a subgroup of the symmetric group* S_n *with generating set* $S = \{\pi_1, \pi_2, \ldots, \pi_k\}$. *For a given partition* $n_1 + n_2 + \cdots + n_r = n$, **the quotient digraph** *is isomorphic to the relative lift* Γ^β *with base digraph a singleton with arcs* $a_1, \ldots, a_k$, *group* G, *relative voltage assignment* β *defined by* $\beta(a_i) = \pi_i$ *for* $i = 1, \ldots, k$, *and stabilizer subgroup* $H = \mathrm{Stab}_G(V_1) \cap \cdots \cap \mathrm{Stab}_G(V_r)$, *where* $V_1 \cup \cdots \cup V_r$ *is a partition of* $[n]$ *with* $|V_i| = n_i$ *for* $i = 1, \ldots, r$.

An example: $P(4)$

From SageMath: the matrices of the irreducible representations of S_4:

- ρ_1 (partition $4 = 4$),
- ρ_2 (partition $4 = 3 + 1$),
- ρ_3 (partition $4 = 2 + 2$),
- ρ_4 (partition $4 = 2 + 1 + 1$),
- ρ_5 (partition $4 = 1 + 1 + 1 + 1$),

related to the permutations $a = 1243$, $b = 1432$, and $c = 4321$.

An example: $P(4)$

From SageMath: the matrices of the irreducible representations of S_4:

- ρ_1 (partition $4 = 4$),
- ρ_2 (partition $4 = 3 + 1$),
- ρ_3 (partition $4 = 2 + 2$),
- ρ_4 (partition $4 = 2 + 1 + 1$),
- ρ_5 (partition $4 = 1 + 1 + 1 + 1$),

related to the permutations $a = 1243$, $b = 1432$, and $c = 4321$.

$\mathrm{sp}\, P(4)$ is the union of the following spectra:

(i) $1 \cdot \mathrm{sp}\, \rho_1(B) = \{[3]^1\}$,

(ii) $3 \cdot \mathrm{sp}\, \rho_2(B) = \{[2]^3, [0]^3, [-1]^3\}$,

(iii) $2 \cdot \mathrm{sp}\, \rho_3(B) = \{[2]^2, [0]^2\}$,

(iv) $3 \cdot \mathrm{sp}\, \rho_4(B) = \{[\frac{-1+\sqrt{17}}{2}]^3, [-2]^3, [\frac{-1-\sqrt{17}}{2}]^3\}$,

(v) $1 \cdot \mathrm{sp}\, \rho_5(B) = \{[-1]^1\}$.

$$\mathrm{sp}\, P(4) = \left\{[3]^1, [2]^5, \left[\frac{-1+\sqrt{17}}{2}\right]^3, [0]^5, \left[\frac{-1-\sqrt{17}}{2}\right]^3, [-1]^4, [-2]^3\right\}$$

Partition: 4=4	a: (1)	b: (1)	c: (1)
$a+b+c$: (3)	Dimension: 1	Eigenvalues: 3	Spectrum: $[3]^1$

Partition: 4=3+1	a: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$	b: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	c: $\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$a+b+c$: $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	Dimension: 3	Eigenvalues: $2, 0, -1$	Spectrum: $[2]^3, [0]^3, [-1]^3$

Partition: 4=2+2	a: $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	b: $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	c: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$a+b+c$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	Dimension: 2	Eigenvalues: $2, 0$	Spectrum: $[2]^2, [0]^2$

Partition: 4=2+1+1	a: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	b: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	c: $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$
$a+b+c$: $\begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix}$	Dimension: 3	Eigenvalues: $\frac{-1+\sqrt{17}}{2}, -2, \frac{-1-\sqrt{17}}{2}$	Spectrum: $[\frac{-1+\sqrt{17}}{2}]^3, [-2]^3, [\frac{-1-\sqrt{17}}{2}]^3$

Partition: 4=1+1+1+1	a: (-1)	b: (-1)	c: (1)
Sum: (-1)	Dimension: 1	Eigenvalues: -1	Spectrum: $[-1]^1$

Spectra of the quotient graphs of $P(4)$

rank	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	Spectrum
rk$\rho(H_1)$	1	0	0	0	0	$\{3\}$
rk$\rho(H_2)$	1	1	0	0	0	$\{3, 2, 0, -1\}$
rk$\rho(H_3)$	1	1	1	0	0	$\{3, 2^2, 0^2, -1\}$
rk$\rho(H_4)$	1	2	1	1	0	$\{3, 2^3, 0^3, \frac{-1\pm\sqrt{17}}{2}, [-1]^2, -2\}$
rk$\rho(H_5)$	1	3	2	3	1	$\{3, 2^5, 0^5, [\frac{-1\pm\sqrt{17}}{2}]^3, 0^5, [-1]^4, [-2]^3\}$
sp $\rho(B)$	$\{3\}$	$\{2, 0, -1\}$	$\{2, 0\}$	$\{\frac{-1\pm\sqrt{17}}{2}, -2\}$	$\{-1\}$	

where

- $\rho_1 : 4 = 4$
- $\rho_2 : 4 = 3 + 1$
- $\rho_3 : 4 = 2 + 2$
- $\rho_4 : 4 = 2 + 1 + 1$
- $\rho_5 : 4 = 1 + 1 + 1 + 1$

References

L. BABAI.
Spectra of Cayley graphs.
J. Combin. Theory Ser. B **27** (1979) 180–189.

E. T. BASKORO, L. BRANKOVIĆ, M. MILLER, J. PLESNÍK, J. RYAN, J. ŠIRÁŇ.
Large digraphs with small diameter: A voltage assignment approach.
JCMCC **24** (1997) 161–176.

M. BURROW.
Representation Theory of Finite Groups.
Dover, New York, 1993.

C. DALFÓ, M. A. FIOL.
Spectra and eigenspaces from regular partitions of Cayley (di)graphs of permutation groups.
Linear Algebra Appl. **597** (2020) 94–112.

C. DALFÓ, M. A. FIOL, M. MILLER, J. RYAN, J. ŠIRÁŇ.
An algebraic approach to lifts of digraphs,
Discrete Appl. Math. **269** (2019) 68–76.

C. DALFÓ, M. A. FIOL, J. ŠIRÁŇ.
The spectra of lifted digraphs.
J. Algebraic Combin. **50** (2019) 419–426.

References

C. Dalfó, M. A. Fiol, S. Pavlíková, J. Širáň.
Spectra and eigenspaces of arbitrary lifts of graphs.
(2019), submitted, http://arxiv.org/abs/1903.10776.

W. H. Gates and C. H. Papadimitriou.
Bounds for sorting by prefix reversal.
Discrete Math. **27** (1979) 47–57.

C. D. Godsil.
Algebraic Combinatorics.
Chapman and Hall, New York, 1993.

J. L. Gross and T. W. Tucker.
Generating all graph coverings by permutation voltage assignments,
Discrete Math. **18** (1977) 273–283.

G. James and M. Liebeck.
Representations and Characters of Groups.
2nd ed., Cambridge Univ. Press, 2001.

L. Lovász.
Spectra of graphs with transitive groups.
Period. Math. Hungar. **6** (1975) 191–196.