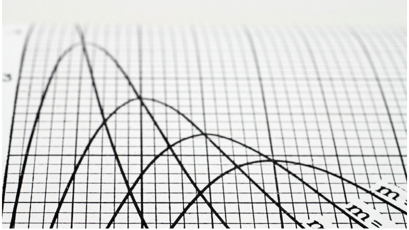
Moments of permutation statistics and central limit theorems



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(joint work with Niraj Khare)

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Background

Consider the set, Π_n, of all partitions of $[n] := \{1, 2, \ldots, n\}$.

Their number is B_n - the n-th Bell number. For example, $B_3 = 5$:

$\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{1\}, \{2, 3\}\}$, $\{1, 2, 3\}$.

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Can we find $M(X_1; n)$, where $M(f; n) := \sum_{\lambda \in \Pi_n} f(\lambda)$?

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Let $cr_2(\lambda) :=$ the number of 2-crossings in $\lambda \in \Pi_n$,
i.e., numbers $i_1 < i_2 < j_1 < j_2$, such that i_1, j_1 and i_2, j_2 are in two different blocks. For instance, $cr_2(\{\{1, 3\}, \{2, 4, 5\}\}) = 2$.
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Can we find $M(cr_2; n)$?
$M(cr_2; n) = \frac{1}{4}(-5B_{n+2} + (2n+9)B_{n+1} + (2n+1)B_n)$ (Kasraoui, 2013 [6]).

Chern, Diaconis, Kane and Rhoades [4] found that

$$M(X_1^2;n) = nB_{n-1} + (n^2 - n)B_{n-2}.$$

and

$$M(cr_2^2;n) = \frac{1}{144}(225B_{n+4} - (180n+814)B_{n+3} + (36n^2+156n+489)B_{n+2} + (72n^2 + 72n - 260)B_{n+1} + (36n^2 + 24n - 23)B_n).$$

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Theorem 1 (CDKR)

For a family of set partition statistics, the moments can be written as linear combinations of shifted Bell numbers, where the coefficients are polynomials in n.

Khare, Lorentz and Yan [7] developed the same approach on the set of perfect matchings (set partitions with blocks of size 2) on $[2m]$.

Theorem 2 (KLY)

For a family of statistics on perfect matchings, the moments can be written as linear combinations of double factorials with constant coefficients.

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For example, they found:

$$\sum_{M \in \mathcal{M}_{2m}} cr_2^2(M) = \binom{2m}{4} T_{2m-4} + 12 \binom{2m}{6} T_{2m-6} + 70 \binom{2m}{8} T_{2m-8},$$

where $T_{2m} = |\mathcal{M}_{2m}| = (2m-1)(2m-3)\cdots 3 \cdot 1 = (2m-1)!!$.

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Goal: Develop the same approach for permutations!

- *permutation - ordering of the numbers in* $[n]$.
- S_n *- the set of permutations of size n.*
  Example: $4172365 \in S_7$.
- *Let* $A(\pi) \coloneqq \{(u,v) \mid u = \pi_i, v = \pi_j, i < j\}$ *be the arc set of* π.
  Example: $A(312) = \{(3,1),(3,2),(1,2)\}$.
- $\mathrm{red}(s_1 s_2 \cdots s_k) \coloneqq p_1 \cdots p_k \in S_k$, *where* $p_i < p_j$ *iff* $s_i < s_j$.
  Example: $\mathrm{red}(6253) = 4132$.

Definition

(i) A *permutation pattern* $\underline{P}$ of size k is a tuple $\underline{P} = (P, \boldsymbol{C}(\underline{P}), \boldsymbol{D}(\underline{P}))$, where $P = p_1 \cdots p_k \in S_k$ and $\boldsymbol{C}(\underline{P}) \subseteq [k-1], \boldsymbol{D}(\underline{P}) \subseteq [k-1]$.

(ii) An *occurrence* of the pattern $\underline{P} = (p_1 p_2 \cdots p_k, \boldsymbol{C}(\underline{P}), \boldsymbol{D}(\underline{P}))$ of size k in $\sigma \in S_n$ is a tuple $t = (t_1, t_2, \ldots, t_k)$ with $t_i \in [n]$, such that:

a) $t_1 < t_2 < \cdots < t_k$.
b) $(t_i, t_j) \in A(\sigma)$, if and only if $(i, j) \in A(P)$.
c) if $i \in \boldsymbol{C}(\underline{P})$, then the positions of t_{p_i} and $t_{p_{i+1}}$ in σ are consecutive.
d) if $i \in \boldsymbol{D}(\underline{P})$, then $t_{i+1} = t_i + 1$.

Write $t \in_{\underline{P}} \sigma$, if t is an occurrence of $\underline{P}$ in σ.

Examples:

1. $\underline{P} = 132 = (132, \emptyset, \emptyset)$ [*classical* patterns].

   $t = (3, 4, 5) \in_{\underline{P}} 31524$, since $\mathrm{red}(354) = 132$.

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Examples:

2. $\underline{P} = \underline{32}14 = (3214, \{1\}, \emptyset)$ [*vincular* patterns].

$t = (2, 3, 5, 7) \in_{\underline{P}} 4536217$, since $\mathrm{red}(5327) = 3214$
and the positions of $t_3 = 5$ and $t_2 = 3$ are consecutive.

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Examples:

3. $\underline{P} = \dfrac{12\overline{34}}{4\underline{31}2} = (4312, \{2\}, \{3\})$ [*bivincular* patterns].

$t = (1, 3, 5, 6) \in_{\underline{P}} 625143$, since $\mathrm{red}(6513) = 4312$,
the positions of $t_3 = 5$ and $t_1 = 1$ are consecutive, and $t_4 = 6 = t_3 + 1$.

Family of statistics

simple statistic: a pattern $\underline{P}$ of size k and a valuation function $Q(t,w) = Q_1(t)Q_2(w)$, where $Q_1, Q_2 \in \mathbb{Z}[y_1, \ldots, y_k, m]$.

$$f(\sigma) = f_{\underline{P},Q}(\sigma) := \sum_{t \in_{\underline{P}} \sigma} Q(t, \sigma^{-1}(t)) = \sum_{t \in_{\underline{P}} \sigma} Q_1(t)Q_2(\sigma^{-1}(t)).$$

f is of degree $d(f) :=$ 2k + deg(Q).

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Examples (simple statistic):

1. $\mathrm{cnt}_{\underline{P}}(\sigma) := f_{\underline{P},1}(\sigma) = \sum_{t \in_{\underline{P}} \sigma} 1$, for any pattern $\underline{P}$,
   e.g., 21, 1324, $\underline{123}$, $\genfrac{}{}{0pt}{}{1\overline{23}}{3\underline{12}}$.

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2. $\operatorname{drops}(\sigma) := \sum_{\sigma_i > \sigma_{i+1}} \sigma_i - \sigma_{i+1} = \sum_{(t_1,t_2) \in_{\underline{21}} \sigma} t_2 - t_1$.
   $\underline{P} = \underline{21}$, $Q(t,w) = Q_1(t)Q_2(w)$, where
   $Q_1(t) = Q_1(t_1,t_2) = t_2 - t_1$ and $Q_2(w) = 1$.

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Example (statistic):

$$\text{peakSqSum}(\sigma) \coloneqq \sum_{\sigma(i-1)<\sigma(i)>\sigma(i+1)} \sigma(i)^2 = \sum_{(t_1,t_2,t_3) \in_{\underline{132}} \sigma} t_3^2 + \sum_{(t_1,t_2,t_3) \in_{\underline{231}} \sigma} t_3^2.$$

This is a sum of the simple statistics $f_{\underline{132},t_3^2}$ and $f_{\underline{231},t_3^2}$.

Theorem 3

Let $f_{\underline{P},Q}$ be a simple statistic of degree m, where $|P| = k$, $|\boldsymbol{C}(\underline{P})| = c$ and $|\boldsymbol{D}(\underline{P})| = d$. Then

$$M(f_{\underline{P},Q}, n) = R(n)(n-k)!,$$

where $R(x)$ is a polynomial of degree no more than $m - c - d$. Equivalently for $n \geq k$,

$$M(f_{\underline{P},Q}, n) = \begin{cases} 0 & n < k \\ \displaystyle\sum_{i=0}^{m-c-d} c_i (n-k+i)! & n \geq k \end{cases},$$

for some constants $c_i \in \mathbb{Q}$.

Examples - first moments formulas

Simple statistics:

1. cnt_{1324}.

$$M(\mathrm{cnt}_{1324}, n) = \frac{1}{24}n! - \frac{1}{6}(n+1)! + \frac{1}{8}(n+2)! - \frac{1}{36}(n+3)! + \frac{1}{576}(n+4)!.$$

In fact, $M(\mathrm{cnt}_P, n) = \frac{1}{k!}\binom{n}{k}n!$ for any classical pattern P of size k.

Express rising factorials in terms of falling factorials to get

$$M(\mathrm{cnt}_P, n) = \frac{1}{k!}\binom{n}{k}n! = \frac{(-1)^k}{k!}n! + \sum_{j=1}^{k-1} \frac{(-1)^{k-j}}{(j!)^2(k-j)!}(n+j)! + \frac{1}{(k!)^2}(n+k)!.$$

2. *Descent drop*.

$$M(\mathrm{drops}, n) = -\frac{1}{2}(n+1)! + \frac{1}{6}(n+2)!.$$

Theorem 4

For any statistic f of degree m, there is a positive integer $L \leq \frac{m}{2}$, such that for all $n \geq L$,

$$M(f, n) = U(n)(n - L)!,$$

where $U(n)$ is a polynomial of degree no more than $m + L$. Equivalently, if $n \geq L$,

$$M(f, n) = \sum_{-L \leq i \leq m} \alpha_i (n + i)!,$$

for some constants $\alpha_i \in \mathbb{Q}$.

Example: *Sum of peak squares*.

$$M(\text{peakSqSum}, n) = (n + 1)! - \frac{5}{4}(n + 2)! + \frac{1}{5}(n + 3)!.$$

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$(1,2,3,4,6) \in_{\underline{15}\,\underline{24}3}$ **5****16****24****3**

Merge of patterns

Let $\underline{P_1}$, $\underline{P_2}$ and $\underline{P_3}$ be patterns of sizes k_1, k_2 and k_3, respectively.

A *merge* of $\underline{P_1}$ and $\underline{P_2}$ onto $\underline{P_3}$ is a pair of increasing functions $m_1 : [k_1] \to [k_3]$ and $m_2 : [k_2] \to [k_3]$ with certain properties.

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Example:
$m_1(1) = 1$, $m_2(1) = 2$,
$m_1(2) = 4$, $m_2(2) = 3$,
$m_1(3) = 5$, $m_2(3) = 4$.

Then $m_1, m_2 : \underline{13}2, \underline{13}2 \to \underline{15}\,\underline{24}3$.

Merge of patterns

Lemma 1

Let $\underline{P_1}$ and $\underline{P_2}$ be two patterns. For any $\sigma \in S_n$, there is a one-to-one correspondence between the following sets.

$$\{(s_1, s_2) : s_1 \in_{\underline{P_1}} \sigma, s_2 \in_{\underline{P_2}} \sigma\} \leftrightarrow \{s_3 \in_{\underline{P_3}} \sigma \mid m_1, m_2 : \underline{P_1}, \underline{P_2} \to \underline{P_3}\}$$

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Using Lemma 1, we prove that the product of two simple statistics is a statistic:

$$f_{\underline{P_1},Q_1}(\sigma) g_{\underline{P_2},Q_2}(\sigma) = \sum_{s_1 \in_{\underline{P_1}} \sigma} Q_1(s_1) Q_1'(\sigma^{-1}(s_1)) \sum_{s_2 \in_{\underline{P_2}} \sigma} Q_2(s_2) Q_2'(\sigma^{-1}(s_2))$$

$$\overset{\text{(by Lemma 1)}}{=} \sum_{\underline{P_3}} \left(\sum_{s_3 \in_{\underline{P_3}} \sigma} \left(\sum_{m_1, m_2 : \underline{P_1}, \underline{P_2} \to \underline{P_3}} Q_{m_1,m_2,Q_1,Q_2}(s_3) Q'_{m_1,m_2,Q_1,Q_2}(\sigma^{-1}(s_3)) \right) \right) = \sum_{\underline{P_3}} f_{\underline{P_3},\tilde{Q}}.$$

Theorem 5

Let f be any statistic of degree m. Then, for any positive integer r, the r-th moment of f is given by

$$M(f^r, n) = \sum_{-I \leq i \leq J} \alpha_i (n+i)!,$$

where I and J are constants that satisfy $-I \geq \frac{-rm}{2}$, $J \leq mr$ and $n \geq I$, and the α_i's are rational constants.

First corollary

Corollary 1

If $\underline{P}$ is a vincular pattern of size k, such that $|\boldsymbol{C}(\underline{P})| = c$, then

$$M(\mathrm{cnt}^r_{\underline{P}}, n) = \sum_{0 \le i \le r(k-c)} \alpha_i (n+i)!,$$

for $n \ge rk$.

Zeilberger [8] showed that if $\underline{P}$ is a classical pattern of size k, then $\mathbb{E}(\mathrm{cnt}^r_{\underline{P}})$ for a random permutation of size n, is a polynomial in n of degree rk. Corollary 1 is a generalization.

Examples - higher moments formulas

1. Second moment of the number of double ascents.

$$M(\mathrm{cnt}^2_{\underline{123}}, n) = -\frac{1}{12}n! - \frac{1}{15}(n+1)! + \frac{1}{36}(n+2)!.$$

2. Second moment of $\mathrm{cnt}_{\substack{1\overline{23}\\3\underline{12}}}$.

$$M(\mathrm{cnt}^2_{\substack{1\overline{23}\\3\underline{12}}}, n) = \tfrac{1}{2}n! - \tfrac{9}{28}(n+1)! + \tfrac{29}{672}(n+2)! + \tfrac{11}{10080}(n+3)! - \tfrac{1}{45360}(n+4)!.$$

Second corollary

Corollary 2

Let $\underline{P}$ be a pattern of size k with $|\boldsymbol{C}(\underline{P})| = c$, $|\boldsymbol{D}(\underline{P})| = d$. Then,

$$M(\mathrm{cnt}^r_{\underline{P}}, n) = \sum_{\tilde{k},\tilde{c},\tilde{d}} w^{(r)}_{\tilde{k},\tilde{c},\tilde{d}} \binom{n-\tilde{c}}{\tilde{k}-\tilde{c}} \binom{n-\tilde{d}}{\tilde{k}-\tilde{d}} (n-k)!,$$

where $w^{(r)}_{\tilde{k},\tilde{c},\tilde{d}}$ is the number of ways to merge r copies of $\underline{P}$ and get a pattern $\underline{P^r}$ of size $\tilde{k}$, with $|C(\underline{P^r})| = \tilde{c}$, $|D(\underline{P^r})| = \tilde{d}$ and where $k \leq \tilde{k} \leq rk$, $c \leq \tilde{c} \leq rc$ and $d \leq \tilde{d} \leq rd$.

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Next goal: Apply Corollary 2 to some simple patterns.

Moments of descents

Theorem 6

Let $\mathrm{des} := \mathrm{cnt}_{\underline{21}}$. *Consider a random permutation in* S_n. *Then, for any* $r \geq 2$,

$$\mathbb{E}(\mathrm{des}^r) = \sum_{m=2}^{\min(n,2r)} \sum_{u=1}^{\lfloor \frac{m}{2} \rfloor} \left(\sum_{w=0}^{m-u} (-1)^w \binom{m-u}{w} (m-u-w)^r \right) \left(\sum_{\substack{q_1+\cdots+q_u=m \\ q_i \geq 2}} \binom{m}{q_1,\ldots,q_u} \right) \frac{\binom{n-(m-u)}{u}}{m!}.$$

Moments of descents

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Sketch of proof:

Let $\underline{P^r}$ be a pattern of size m, obtained after a merge of r copies of $\underline{21}$.

Every such $\underline{P^r}$ with u segments has $|\boldsymbol{C}(\underline{P^r})| = m - u$.

Thus $M(\mathrm{cnt}_{\underline{P^r}}, n) = \frac{\binom{n-(m-u)}{u}}{m!} n!$ and

$$w^{(r)}_{\tilde{k},\tilde{c}} = w^{(r)}_{m,m-u} = \sum_{u=1}^{\lfloor \frac{m}{2} \rfloor} \left(\sum_{w=0}^{m-u} (-1)^w \binom{m-u}{w} (m-u-w)^r \right) \sum_{\substack{q_1+\cdots+q_u=m \\ q_i \geq 2}} \binom{m}{q_1,\ldots,q_u}.$$

Moments of minimal descents

Theorem 7

Let $\mathrm{adj} := \mathrm{cnt}_{\overline{12}.\underline{21}}$*. Consider a random permutation in* S_n*. Then, for any* $r \geq 1$,

$$\mathbb{E}(\mathrm{adj}^r) = \sum_{m=2}^{min(n,2r)} \sum_{u=1}^{\lfloor \frac{m}{2} \rfloor} \left(\left(\sum_{w=0}^{m-u} (-1)^w \binom{m-u}{w} (m-u-w)^r \right) \binom{m-u-1}{u-1} u! \frac{\binom{n-(m-u)}{u}^2}{n_{(m)}} \right).$$

We will use this result to prove a limit theorem for adj.

Central limit theorems for $\mathrm{cnt}_{\underline{P}}$

The asymptotic distribution of $\mathrm{cnt}_{\underline{P}}(\sigma)$, when σ is chosen uniformly at random from S_n, is Normal:

i. True, when $\underline{P}$ is a classical pattern (Bóna, 2007 [1]) .

ii. True, when $\underline{P}$ is a vincular pattern (Hofer, 2017 [5])

iii. Not true for an arbitrary bivincular pattern
(last proof by Corteel, Louchard and Pemantle, 2004 [3])

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iii. Not true for an arbitrary bivincular pattern
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We will reprove **i.** and **iii.** and give a lemma that would imply **ii.**

Bóna [1] uses the method of *dependency graphs* to obtain that the following theorem implies the CLT for an arbitrary classical pattern.

Theorem 8

Let $X_n := \mathrm{cnt}_\sigma$ *be the number of occurrences of a classical pattern* $\sigma \in S_k$ *in a random permutation of size n. Then, there exists* $c > 0$*, such that for all n,*

$$\mathrm{Var}(X_n) \geq cn^{2k-1}.$$

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Sketch of proof:

Use Corollary 2 to obtain that

$$\mathrm{Var}(X_n) = \mathbb{E}(X_n^2) - \mathbb{E}^2(X_n) = [a_\sigma(2k)\frac{\binom{n}{2k}}{(2k)!} + a_\sigma(2k-1)\frac{\binom{n}{2k-1}}{(2k-1)!} + \mathcal{O}(n^{2k-2})] - \frac{\binom{n}{k}^2}{(k!)^2},$$

where $a_\sigma(r)$ is the number of ways to merge two copies of σ and get a pattern of size r.

Sketch of proof (cont.)

Note that $a_\sigma(2k) = \binom{2k}{k}^2$ and simplify to get:

$$\mathrm{Var}(X_n) \geq cn^{2k-1} \iff a_\sigma(2k-1) > \binom{2k-1}{k}^2.$$

Sketch of proof (cont.)

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Lemma 2 (Burstein and Hästö, [2, Lemma 4.3])

For any classical pattern $\sigma \in S_k$,

$$a_\sigma(2k-1) > \binom{2k-1}{k}^2.$$

Interpretation of the lemma

$$A_{\sigma,\sigma'}(2k-1) := \{(\pi, x, y) \mid \pi \in S_{2k-1},\ x, y \in \mathrm{subs}(\pi),\ \mathrm{red}(x){=}\sigma,\ \mathrm{red}(y){=}\sigma',\ |x \cap y| = 1\},$$

where $\mathrm{subs}(\pi)$ denotes the set of the subsequences of the permutation π.

$a_\sigma(2k-1)$, is the number of triples in the set $A_{\sigma,\sigma}(2k-1)$.

Example: $A_{312,312}(5)$ contains $(54213, 523, 413)$.

5	4	2	1	3
5		2		3
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Theorem 9

If $a_{\sigma,\sigma'}(2k-1) := |A_{\sigma,\sigma'}(2k-1)|$, then Lemma 2 is equivalent to

$$a_\sigma(2k-1) > \mathbb{E}(a_{\sigma,\sigma'}(2k-1)),$$

for any fixed $\sigma \in S_k$ and σ' chosen uniformly at random in S_k.

CLT for vincular patterns

Theorem 10 (Hofer, [5])

Let $X_n = \mathrm{cnt}_{\underline{\sigma}}$ *be the number of occurrences of a vincular pattern* $\underline{\sigma}$ *with* j *blocks, in a random permutation of size* n. *Then, there exists* $c > 0$, *such that for all* n,

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Theorem 10 (Hofer, [5])

Let $X_n = \mathrm{cnt}_{\underline{\sigma}}$ be the number of occurrences of a vincular pattern $\underline{\sigma}$ with j blocks, in a random permutation of size n. Then, there exists $c > 0$, such that for all n,

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Let $b_{\underline{\sigma}}(m, j')$ be the number of merges of two copies of $\underline{\sigma}$, where the resulting pattern is of size m and has j' blocks.

Example: $\underline{\sigma} = \underline{431}\,\underline{52}$. Below is a merge of two copies of σ.
The resulting pattern, $\underline{6531}\,\underline{84}\,\underline{72}$, has size $m = 8$ and $j' = 3$ blocks:

6	5	3	1	8	4	7	2
6	5	3		8	4		
	5	3	1			7	2

Merge of two copies of the pattern $\underline{431}\,\underline{52}$.

Theorem 10 (Hofer, [5])

Let $X_n = \mathrm{cnt}_{\underline{\sigma}}$ be the number of occurrences of a vincular pattern $\underline{\sigma}$ with j blocks, in a random permutation of size n. Then, there exists $c > 0$, such that for all n,

$$\mathrm{Var}(X_n) \geq cn^{2j-1}.$$

Theorem 11

Theorem 10 is equivalent to

$$\sum_{l=1}^{M_{\underline{\sigma}}} (2k)_l b_{\underline{\sigma}}(2k-l, 2j-1) > \binom{2k}{k}\binom{2j-1}{j} j,$$

for any vincular pattern $\underline{\sigma}$ with j blocks, where $M_{\underline{\sigma}}$ is the maximal size of a block of $\underline{\sigma}$.

Recall that adj denotes $\mathrm{cnt}_{\substack{\overline{12}\\ \underline{21}}}$.

We use Theorem 7 and the method of moments to prove the following.

Theorem 12

adj *converges in distribution to* $\mathrm{Po}(1)$.

<u>Sketch of proof:</u>

Show that $\lim_{n\to\infty} \mathbb{E}(\mathrm{adj}^r) = B_r$, where B_r is the r-th Bell number.

It is well-known that the r-th moment of $\mathrm{Po}(1)$ variable is B_r.

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It is well-known that the r-th moment of $\mathrm{Po}(1)$ variable is B_r.

Therefore, $\mathrm{cnt}_{\underline{P}}$ does not necessarily converge to a Normal distribution, when $\underline{P}$ is a bivincular pattern.

Further questions

1) Can we find a combinatorial proof of Lemma 2 and the corresponding fact for vincular patterns?

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   Back in 2004, Zeilberger suggested some other structures in the abstract of [8]:

   "…This would be hopefully followed by sequels applied to other combinatorial objects like graph-colorings, Boolean functions, and Random Walks…"

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Thanks for the attention!

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