Maximum genus embeddings of eulerian graphs and digraphs with specified faces

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General goals

We will discuss two related ideas.

(1) Embeddings with euler circuit faces, for eulerian graphs and digraphs.

(2) Maximum genus directed embeddings, for eulerian digraphs.

In the orientable case these two ideas are closely linked. In the nonorientable case they are not so closely linked. Will look mostly at orientable case, with nonorientable case at end (if time).

We also consider situations where we have some pre-specified faces (relative embeddings). In particular, we consider pre-specified faces forming a circuit decomposition of an eulerian graph or digraph.

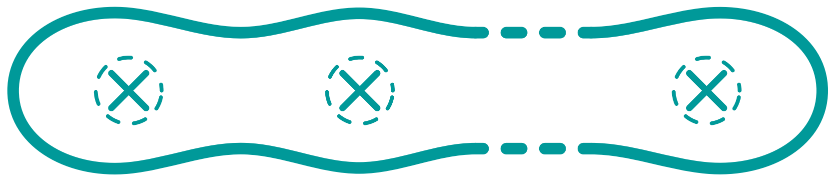
Main results give existence of certain embeddings based on congruence condition for vertex degrees or on density condition.

Surfaces and embeddings

Surfaces: compact, connected;

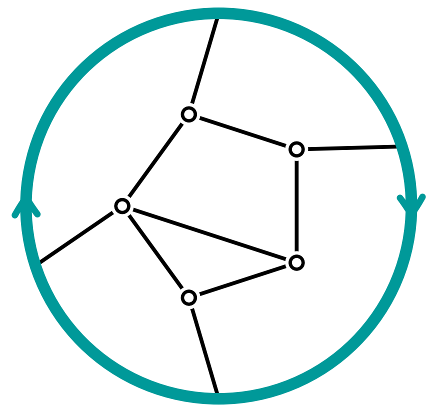
S_h orientable

N_k nonorientable



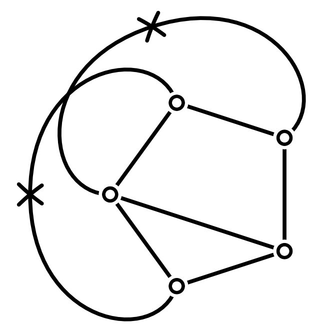
Embedding: Draw graph in surface without edge crossings.

projective plane N_1



Cellular embedding: every face homeomorphic to an open disk.

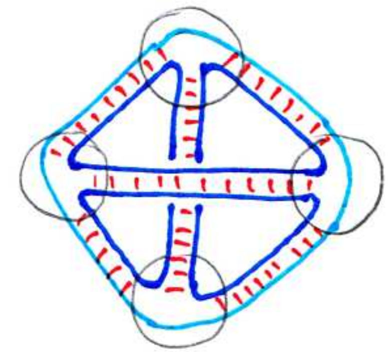
Rotation scheme describes cellular embedding by cyclic order of edges at each vertex and by which edges are twisted (shown as X; left/right swap as you move along edge).



Can also describe by listing facial boundary walks.

Motivation

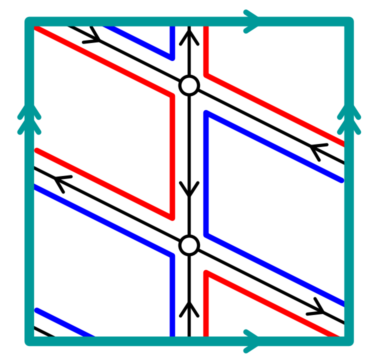
Circuit = closed trail, may repeat vertices but not edges.



(1) DNA models of graphs correspond to orientable embeddings.

Good: single strand of DNA that covers all edges: reporter strand. Better: euler circuit strand. Even better: two euler circuit strands.

(2) Directed embedding of a digraph: each face is bounded by a directed closed walk in the digraph (not just a closed walk in the underlying graph). Digraph must be eulerian.



Directed orientable embeddings with two faces bounded by directed euler circuits would have minimum number of faces, so maximum genus.

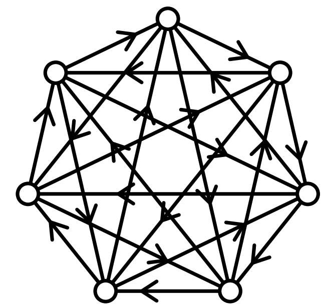
Orientable directed embeddings have 2-face-colouring: boundary direction clockwise for profaces, anticlockwise for antifaces. Conversely, any 2-face-colourable directed embedding must be orientable.

Will see that we can sometimes specify the proface(s) and find a single euler circuit antiface.

Previous results: bi-eulerian embeddings

Bi-eulerian (directed) embedding: has two euler circuit faces.

Edmonds, 1965: (2) Every eulerian graph has a bi-eulerian embedding. But no control over whether orientable or nonorientable.



Bonnington, Conder, Morton & McKenna (BCMM), 2002: Let D be an eulerian tournament (orientation of complete graph) and $n = |V(D)|$ (which must be odd). Let T be a directed euler circuit in D. Then D has an orientable directed embedding with T as the only proface and

(a) one euler circuit antiface, if $n \equiv 3 \pmod 4$ (bi-eulerian embedding), or
(b) two antifaces, if $n \equiv 1 \pmod 4$.

So (a) actually achieves something stronger: bi-eulerian embedding with one of the euler circuits specified in advance. But eulerian tournaments are a fairly special case.

We will give two results that generalize (a) in different ways, one of which also generalizes (b).

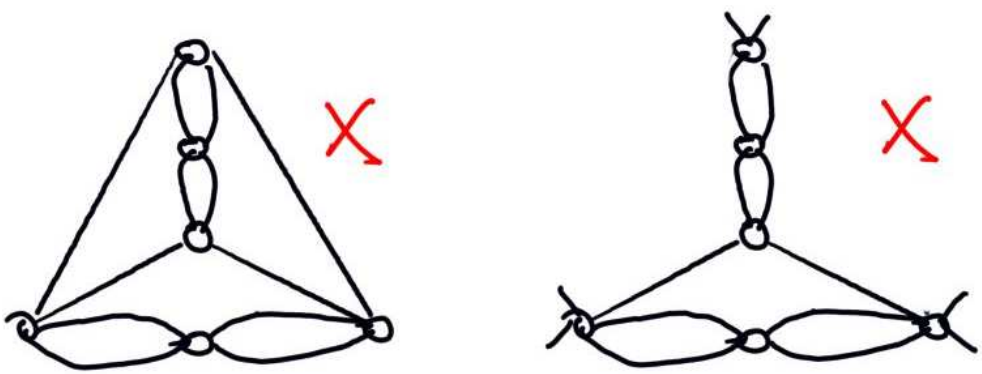
Necessary but not sufficient conditions

Necessary conditions for an orientable bi-eulerian embedding (either graph or digraph):

(1) The number of vertices of degree 0 mod 4 must be even. (From Euler's formula.)

(2) There is no bad 2-edge-cut, separating two subgraphs of G each with an odd number of vertices of degree 0 mod 4 in G. (By looking at 2-edge-cut reduction process.)

Obvious question: Are these two necessary conditions sufficient?



Answer: No. Counterexample with 12 edges, found by computer. (Karen Collins pointed out that a certain structure in our counterexample always rules out an orientable bi-eulerian embedding, and we generalized to a family of structures.)

Main result based on congruence condition

E & E-M, 2024: Suppose D is an eulerian digraph in which every vertex has (total) degree that is 2 mod 4. Let T be a directed euler circuit in D. Then D has a bi-eulerian directed embedding with T as the proface.

Part (a) of Bonnington et al.'s result is a special case.

Proof idea: Actually have three different proofs.

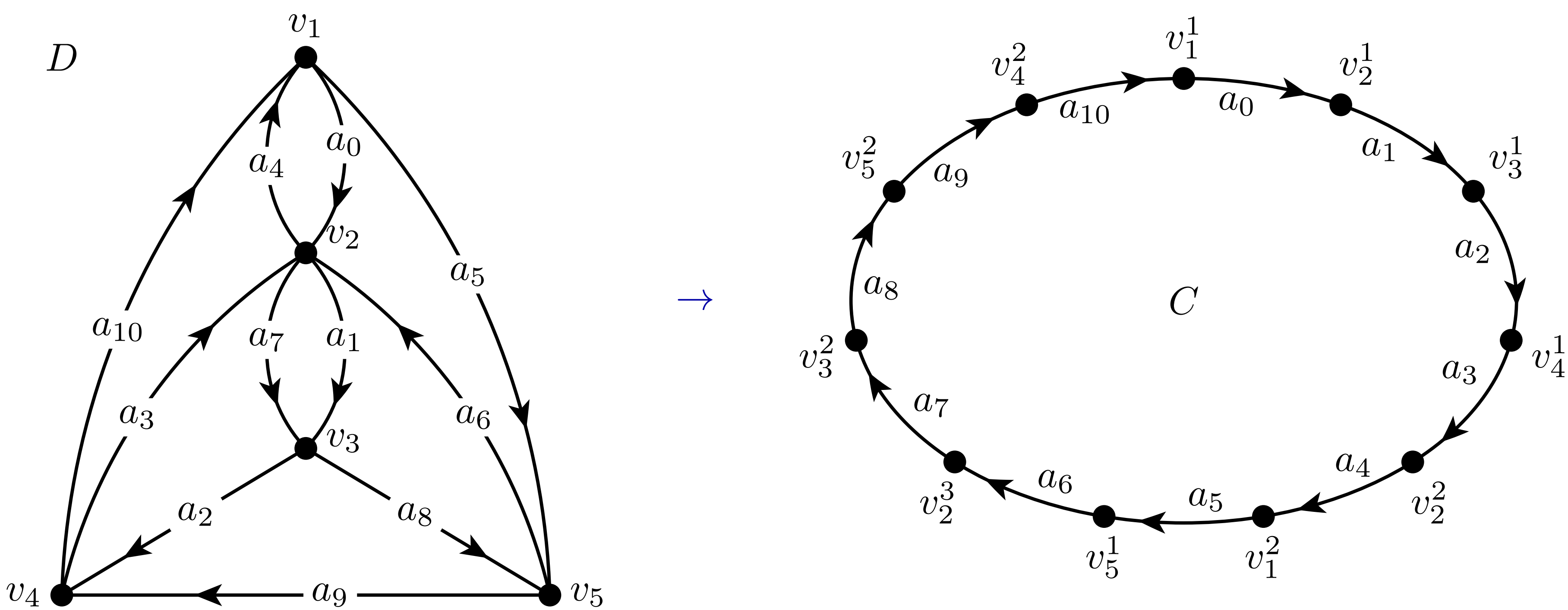
(1) Take $T' = T$ and modify T' one vertex at a time until it is the required second euler circuit face. Based on ideas of Fleischner, 1990 for finding compatible euler circuits.

(2) Use results of Bonnington, 1994 on maximum genus relative embeddings.

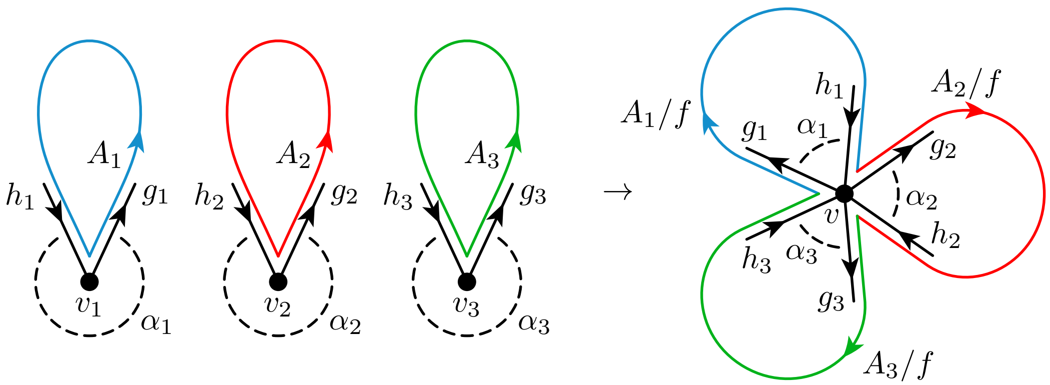
(3) Adapt ideas for undirected eulerian graphs due to Škoviera and Nedela, 1990 based on previous work by Glukhov, 1977 and Javors'kiĭ, 1973. Idea is to pull vertices of the digraph apart until we have a directed cycle, which has a bi-eulerian directed embedding, then gradually glue vertices back together, three at a time, maintaining directed bi-eulerian embedding.

In final version we use approach (3) since it requires less machinery and is easier to extend to other situations.

Pulling an eulerian digraph apart

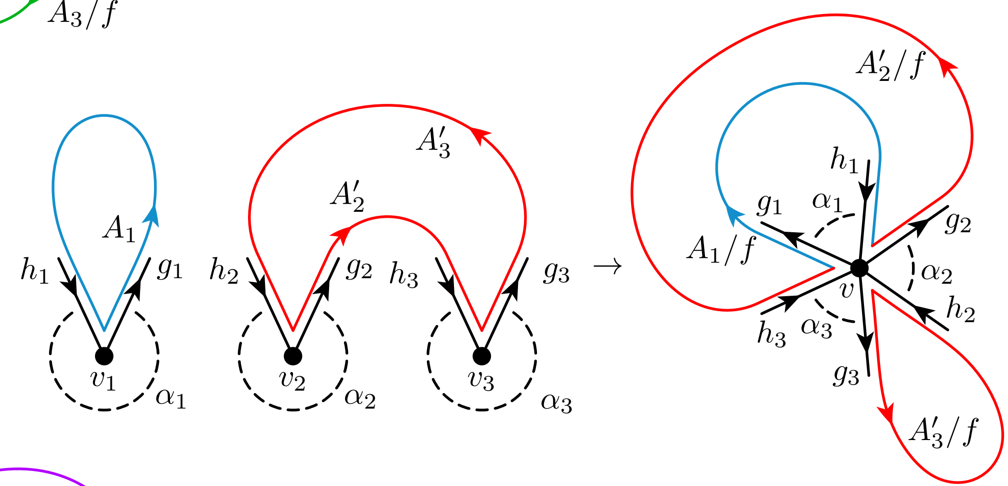


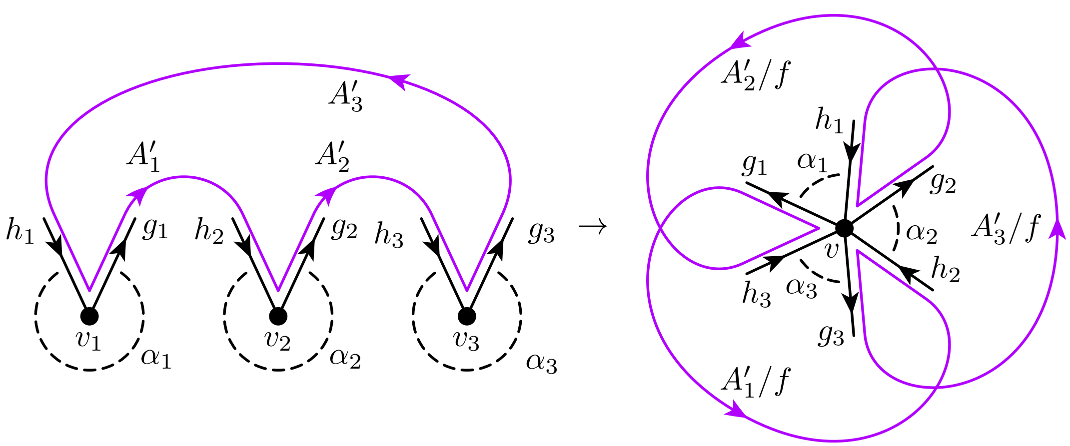
Gluing an eulerian digraph together, three vertices at a time



Combining three faces

Keeping two faces



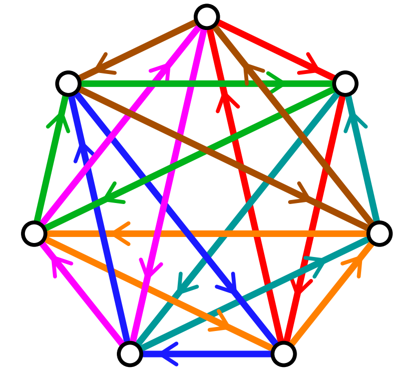


Keeping one face

Previous results: embedding designs

Some design theory structures can be represented as (directed) triangle decompositions. Want maximum genus (directed) embeddings.

Griggs, McCourt & Širáň (GMS), 2020: Decompose the edges of K_n into triangles (a Steiner triple system), and orient each triangle, giving digraph D. Then there is an orientable directed embedding of D whose profaces are the oriented triangles, with one euler circuit antiface.



Griggs, McCourt & Širáň (GMS), 2020: Decompose the edges of $K_{p,p,p}$ (p odd) into triangles (a latin square of odd order), and orient each triangle, giving digraph D. Then there is an orientable directed embedding of D whose profaces are the oriented triangles, with one euler circuit face.

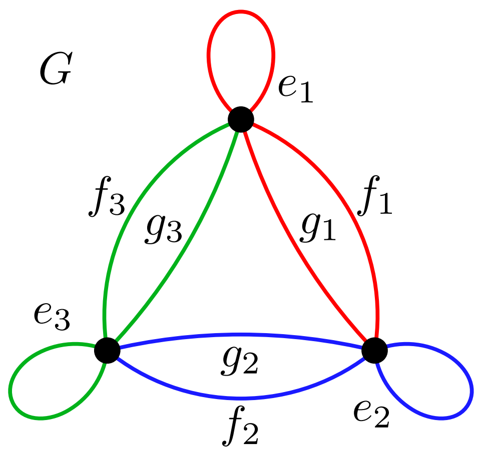
Undirected versions of above results were earlier given by Grannell, Griggs & Širáň, 2005 and by Griggs, Psomas & Širáň, 2018, including nonorientable counterparts.

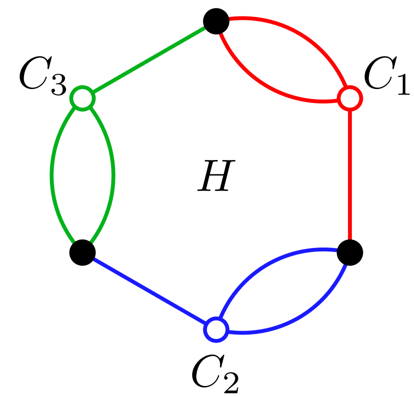
Motivates stronger goal: Given a decomposition $\mathcal{C}$ of the arcs of an eulerian digraph into directed circuits, find a directed embedding in which $\mathcal{C}$ is the set of profaces, with one euler circuit antiface to complete the embedding.

Circuit decompositions that cannot be completed

Erskine, Griggs & Širáň, 2020: If n is odd and $n \geq 21$ there is a 6-regular n-vertex simple graph with a decomposition $\mathcal{T}$ into triangles which cannot be completed to an orientable embedding by adding an euler circuit face. Can then derive digraph examples. Motivation was looking for maximum genus embeddings of configurations n_3.

Smaller examples can be found if loops and multiple edges are allowed.





Consider $\mathcal{T} = \{C_1, C_2, C_3\}$ with C_i having edges e_i, f_i, g_i. There is no orientable embedding of G with elements of $\mathcal{T}$ and an euler circuit as faces, because vertex/circuit incidence graph H has no 1-face orientable embedding.

Main result for dense digraphs

E & E-M, 2024: Suppose D is an n-vertex eulerian digraph (loops and multiple arcs allowed) in which each vertex has at least $(4n+2)/5$ neighbours. Let $\mathcal{C}$ be a directed circuit decomposition of D. Then D has an orientable directed embedding with $\mathcal{C}$ as the set of profaces and with one or two antifaces.

- Very general since $\mathcal{C}$ can be any circuit decomposition (not just euler circuit or triangle decomposition), D only has to satisfy simple density condition (no special structure).
- Euler characteristic must be even, so we may not be able to have just one antiface. But if there is exactly one antiface, it is an euler circuit.
- Implies BCMM tournament result and GMS Steiner triple system result (also a tournament result).
- Does not imply GMS latin square result (would need bound of $2n/3$ or smaller, instead of $(4n+2)/5$).

Proof outline for dense graphs result

E & E-M, 2024: Suppose D is an n-vertex eulerian digraph (loops and multiple arcs allowed) in which each vertex has at least $(4n + 2)/5$ neighbours. Let $\mathcal{C}$ be a directed circuit decomposition of D. Then D has an orientable directed embedding with $\mathcal{C}$ as the set of profaces and with one or two antifaces.

Proof ideas: Extend and simplify proof of BCMM.

Start with arbitrary embedding in which $\mathcal{C}$ is set of profaces (easy to find), then modify antifaces as follows.

(1) If three antifaces meet at a vertex, reduce them to a single antiface.

(2) If have interlacing situation, recombine antifaces so can then apply (1).

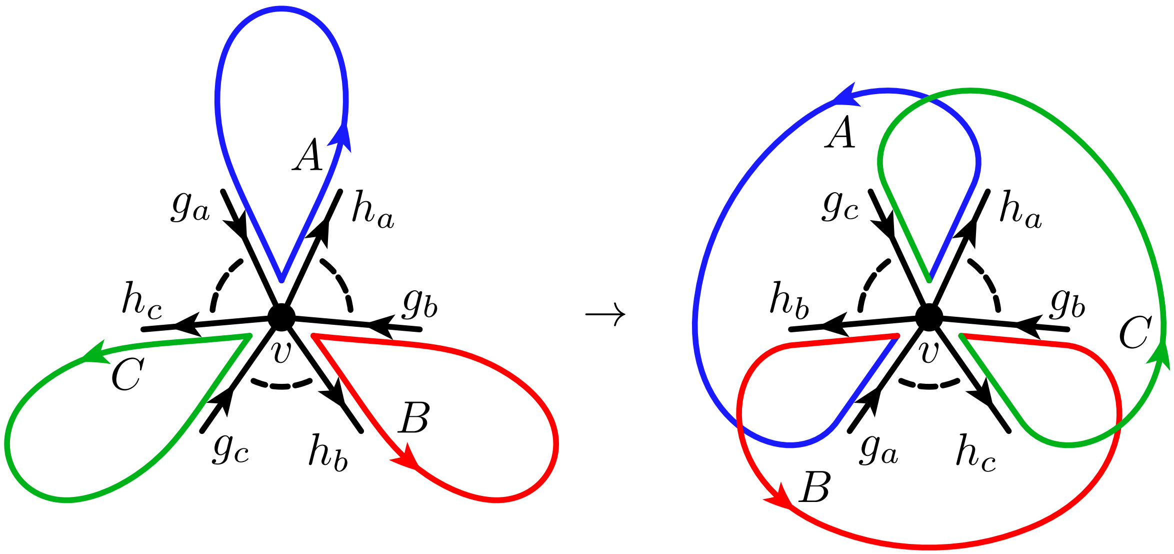
(3) If neither (1) nor (2) apply, try to blow up small antiface by recombining with large antiface to create two moderately large antifaces. Increases chances of being able to apply (1) or (2).

Continue until we have just one or two antifaces.

(1) Reduction: merging three antifaces

If we have three distinct antifaces incident with a vertex v (v is reducible), we can merge them by changing the rotation at v.

Standard operation in undirected case, going back at least to Ringel, 1961.



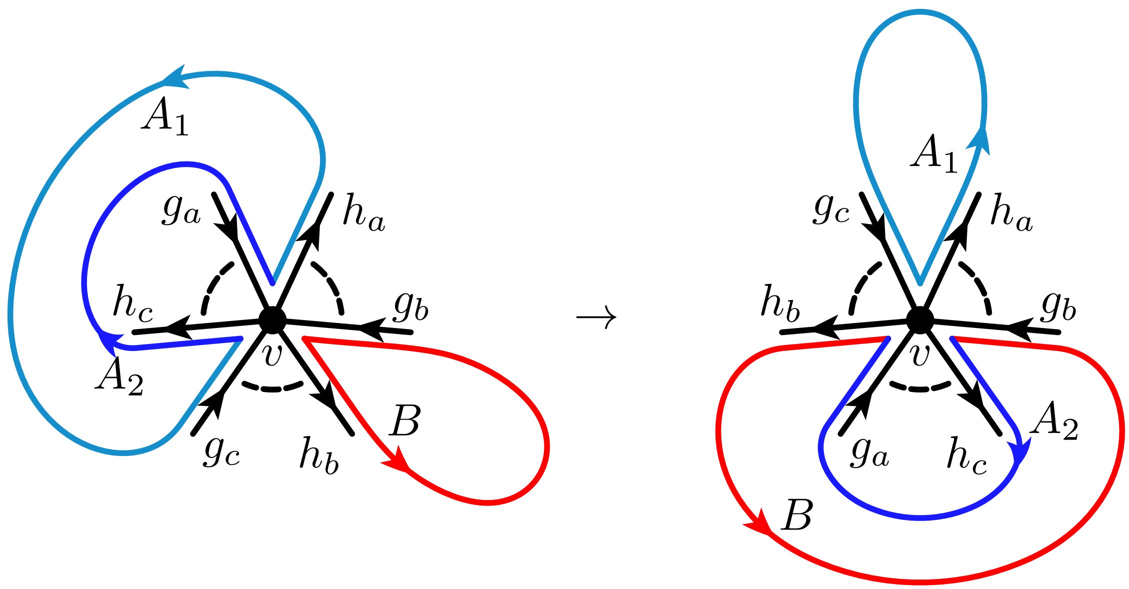
Repeat until no reducible vertices, at most two distinct antifaces incident with each vertex: locally irreducible embedding.

BCMM do something similar but more complicated to get locally irreducible embedding.

(2) & (3) Recombination: reconfiguring two antifaces

Question: A locally irreducible embedding may still have many antifaces. How do we make progress then? (Cannot combine just two faces: parity issue.)

Lemma: If we have two distinct antifaces A, B incident with a vertex v, where A visits v at least twice, then we can modify the rotation at v to replace A and B by A', B' where $E(B) \subseteq E(B')$, and both $E(A')$ and $E(B')$ intersect $E(A)$.



While recombination does not decrease the number of antifaces, it may create a reducible vertex or help in other ways.

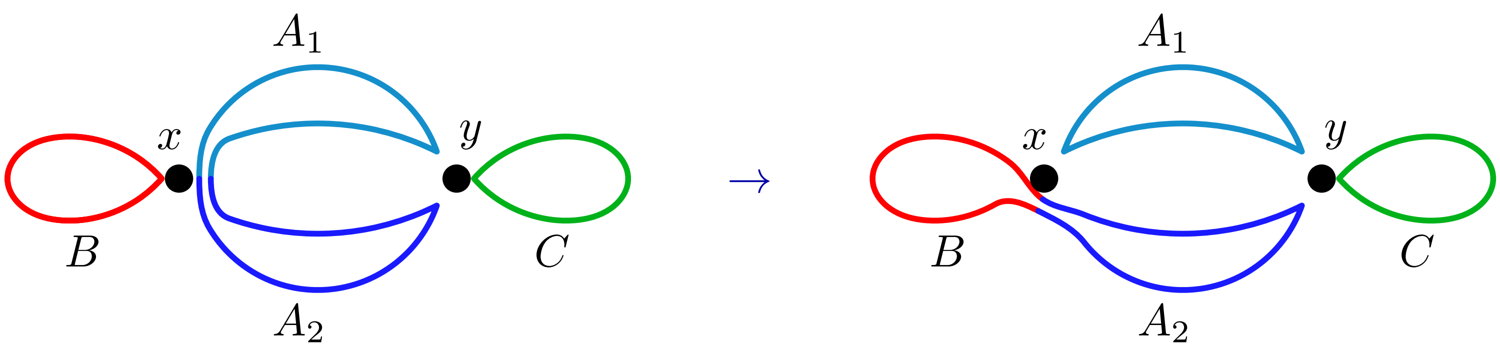
(2) Interlacing $\Rightarrow$ recombine then reduce

Question: How can we ensure that recombination produces a reducible vertex?

Vertices x and y are interlaced on face A if we have $A = (x \ldots y \ldots x \ldots y \ldots)$.

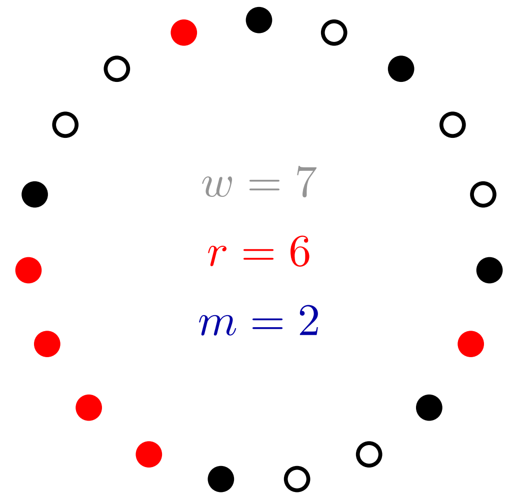
BCMM: If x and y are interlaced on A, and B appears at x while C appears at y, then we can modify the rotation systems at x and y to merge A, B, and C.

Our proof: Recombining A and B at x makes y a reducible vertex.



[Picture does not accurately represent rotation systems.]

(3) Blowing up faces by recombination



Fair Division Lemma: Suppose b black, w white and r red balls are arranged in a circle so that

(a) between every two consecutive black balls there are at most m white balls, and

(b) $r < w$.

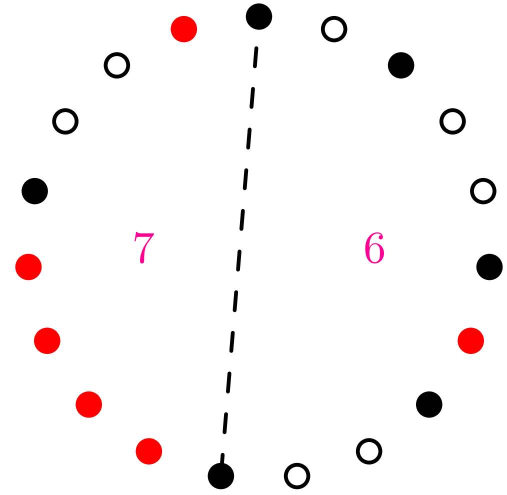
Then there are two black balls so that the number of red or white balls between them (in either direction) lies in the open interval $((r+w)/2 - m, (r+w)/2 + m)$.

Surprising because no restriction at all on the location of the red balls.

Blow Up Lemma: Want to recombine large face A and small face B at common vertex v so vertices of A get shared evenly between the two new faces.

- Circle $\leftrightarrow$ traversal of A, black balls $\leftrightarrow$ occurrences of v, white balls $\leftrightarrow$ one occurrence of each A-neighbour of v (so $m = 2$), and red balls $\leftrightarrow$ one occurrence of each remaining vertex of A.
- Fair Division Lemma indicates which two occurrences of v on A to use.

(3) Blowing up faces by recombination



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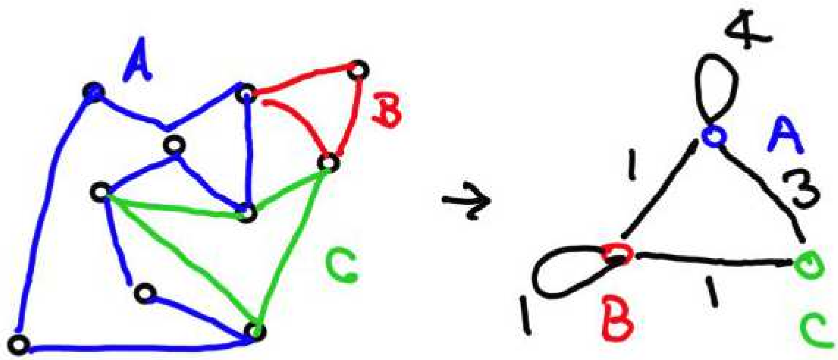
Other proof ingredients

- Three Neighbour Lemma, Diamond Lemma: structures that force interlacing.
- Bipartite Degeneracy Lemma: If a bipartite graph has too many edges it cannot be k-degenerate, so must have a subgraph with minimum degree $k+1$. Use with $k=2$ to apply Three Neighbour Lemma.
- Touch graph K shows how many times (i.e., at how many vertices) the antifaces touch each other. (Assuming embedding is locally irreducible, so at most two antifaces touch at each vertex.)

  Vertex of K $\leftrightarrow$ face of embedding.

  Edge of K records how many vertices see just one face (loop) or a pair of faces (nonloop).

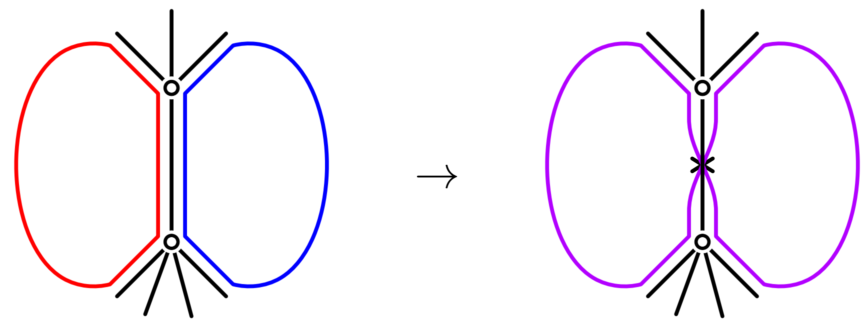


The main proof is divided into cases according to the structure of K.

Nonorientable 1-face (maximum genus) embeddings

Edmonds, 1965: (1) Every connected graph has a 1-face embedding. But no control over whether orientable or nonorientable.

Ringel, 1977 (also Stahl, 1978; Xuong, 1979): Every connected graph has a nonorientable 1-face embedding, unless it is a tree. All embeddings of a tree are planar, hence orientable.



Proof idea: Combine faces by twisting an edge between two faces.

Even nicer result for eulerian graphs: motivated by polypeptide (protein-like) self-assembly problem, where paired strands prefer to go in same direction (unlike DNA).

Fijavž, Pisanski & Rus, 2014: Every eulerian graph has a 1-face embedding in which the facial walk uses every edge twice in the same direction. As far from orientable as possible: the facial walks in an orientable embedding use every edge once in each direction.

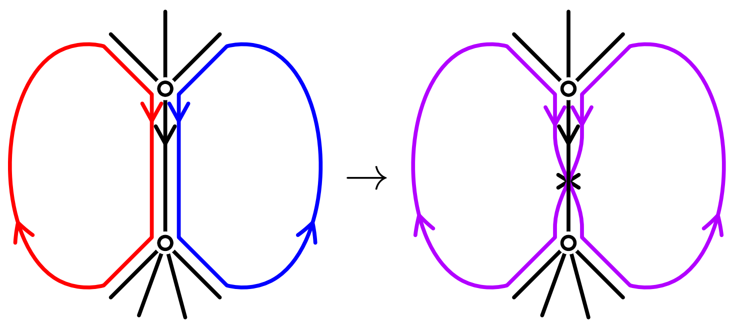
Proof idea: Set up embedding in pseudosurface, resolve pinchpoints (like Edmonds' proof).

Nonorientable 1-face (maximum genus) directed embeddings

Fijavž, Pisanski & Rus, 2014: Every eulerian graph has a 1-face embedding in which the facial walk uses every edge twice in the same direction.

Fijavž, Pisanski & Rus, 2014, restated: For every eulerian graph G, some eulerian orientation D of G has a nonorientable 1-face directed embedding.

E & E-M, 2024: Every eulerian digraph has a nonorientable 1-face directed embedding.



Our result, restated: For every eulerian graph G, every eulerian orientation D of G has a nonorientable 1-face directed embedding.

Proof idea: Twisting edges to combine faces also works for directed embeddings (similar to proof of Ringel, 1977 for undirected graphs).

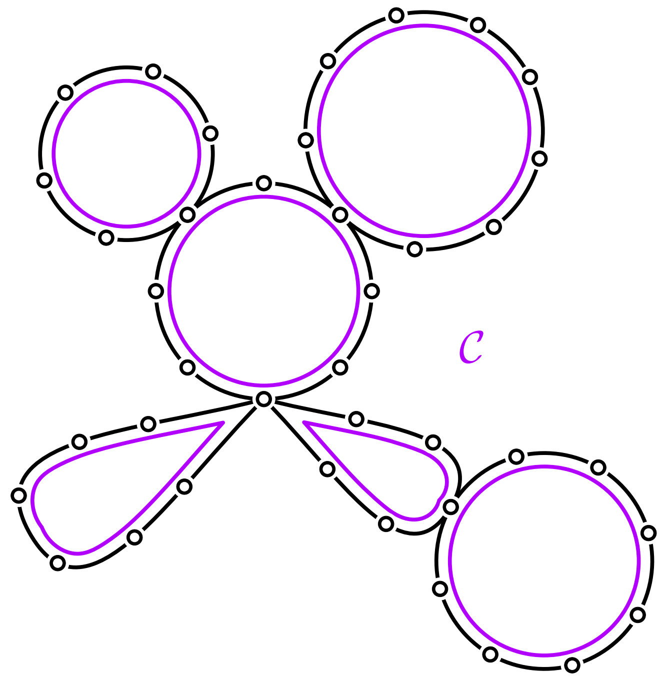
Nonorientable bi-eulerian embeddings, and a generalization

Edmonds, 1965: (2) Every eulerian graph has a bi-eulerian embedding.

E & E-M, 2024: Every eulerian graph has a nonorientable bi-eulerian embedding, unless it is a cycle. The only bi-eulerian embedding of a cycle is planar, hence orientable.

Can specify one of euler circuit faces in advance. Special case of more general result.

E & E-M, 2024: Given a decomposition $\mathcal{C}$ of eulerian G into circuits (closed trails), there is an euler circuit T and an embedding of G with face set $\mathcal{C} \cup \{T\}$. The embedding can be chosen to be nonorientable unless G is a tree of cycles and $\mathcal{C}$ is the set of cycles of G.



Proof idea: Use result of Širáň and Škoviera, 1988 which describes when a partial collection of faces that does not already force nonorientability can be completed to a nonorientable embedding.

Future work

- Maximum genus problem for directed embeddings: Determine the maximum genus of an orientable directed embedding of a given eulerian digraph D.
  - Find analogues of Xuong and Nebeský formulas for maximum orientable genus of graphs.
  - Find analogue of Furst, Gross & McGeoch polynomial-time algorithm for maximum orientable genus of graphs (based on matroid parity algorithm). Can be done for 4-regular eulerian digraphs using delta-matroid parity algorithm of Geelen, Iwata & Murota.
- Sufficient conditions: Extend conditions under which we can find (a) orientable bi-eulerian directed embeddings, or even (b) directed embeddings with specified profaces and one euler circuit antiface.
  - It would be nice to obtain a general result strong enough to cover the GMS latin square result.
  - Perhaps some small constant minimum degree is enough if the digraph is 4-edge-connected. But 4 is not enough for (a) and 6 is not enough for (b).

Thank you!