Small 3-regular graphs and 3-uniform hypergraphs of given girth

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The cage problem

What is the smallest possible graph where each vertex has degree d and the girth (smallest cycle) is g?

A graph achieving the minimum order for a given pair (d, g) is called a *cage*.

If $g = 2k + 1$, all paths of length k or less from a given vertex have distinct endpoints.

For $d \geq 3$:

$$n(d,g) \geq M(d,g) = \begin{cases} \dfrac{d(d-1)^{(g-1)/2} - 2}{d-2}, & g \text{ odd;} \\ \dfrac{2(d-1)^{g/2} - 2}{d-2}, & g \text{ even.} \end{cases}$$

The problem with the cage problem

Even in the simplest non-trivial case of $d = 3$, as the girth increases very few cages are known.

g	Bound	Best	%	g	Bound	Best	%
3	4	4*	100.0	18	1022	2560	250.5
4	6	6*	100.0	19	1534	4324	281.9
5	10	10*	100.0	20	2046	5376	262.8
6	14	14*	100.0	21	3070	16028	522.1
7	22	24*	109.1	22	4094	16206	395.8
8	30	30*	100.0	23	6142	49326	803.1
9	46	58*	126.1	24	8190	49608	605.7
10	62	70*	112.9	25	12286	108906	886.4
11	94	112*	119.1	26	16382	109200	666.6
12	126	126*	100.0	27	24574	285852	1163.2
13	190	272	143.2	28	32766	415104	1266.9
14	254	384	151.2	29	49150	1141484	2322.4
15	382	620	162.3	30	65534	1143408	1744.8
16	510	960	188.2	31	98302	3649794	3712.8
17	766	2176	284.1	32	131070	3650304	2785.0

Table 1: Smallest known cubic graphs. Source: `http://combinatoricswiki.org/`

Hypergraphs

A *hypergraph* $H = (V, E)$ is a set V of vertices and a set E of subsets of V (*hyperedges*).

If $|e| = 2$ for all $e \in E$, this is just a (simple) graph.

If every $v \in V$ lies in precisely d hyperedges, we say H is d-*regular*.

If every $e \in E$ has cardinality r, we say H is r-*uniform*.

So a cubic graph can be thought of as a 3-regular, 2-uniform hypergraph.

A d-regular, r-uniform hypergraph with n vertices and m hyperedges satisfies the equality:

$$nd = mr.$$

Cycles in hypergraphs

A *Berge cycle* of length k in a hypergraph is a sequence $v_0, e_0, v_1, e_1, \ldots, v_{k-1}, e_{k-1}, v_0$ such that each v_i is contained in e_{i-1} and e_i (mod k), all v_i are unique **and** all e_i are unique.

The *girth* of a hypergraph is the length of its smallest Berge cycle.

A hypergraph is *linear* if two distinct hyperedges meet in at most one vertex.

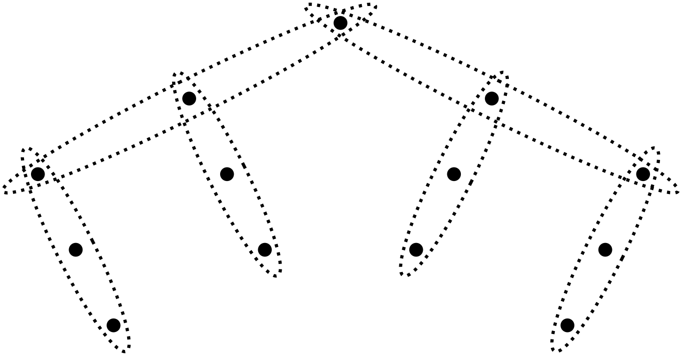
A hypergraph is linear if and only if its girth is at least 3.

Other notions of cycle and girth are available.

Hypergraph Moore bounds

Let H be a d-regular, r-uniform hypergraph of girth $g = 2k + 1$.

From a given vertex, all vertices at distance k or less are distinct. **Example** $d = 2$, $r = 3$, $k = 2$:



$$|V(H)| \geq M(d, r, k) = 1 + d(r-1)\frac{(d-1)^k(r-1)^k - 1}{(d-1)(r-1) - 1}$$

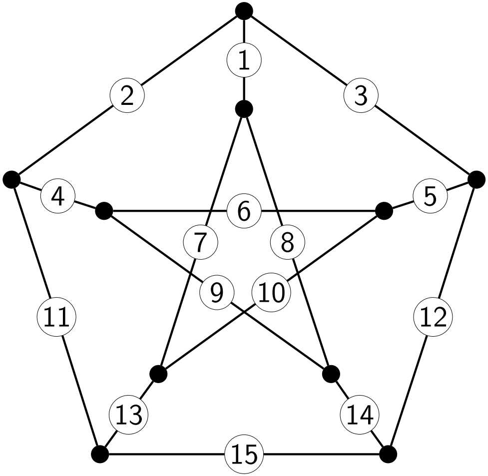
A first observation

For a 2-regular, 3-uniform hypergraph of girth 5 the Moore bound is 13. We can get a hypergraph of order 15 as follows.

Label the edges of the Petersen graph from 1 to 15. These are the vertices of the hypergraph.

The hyperedges are the labels of the edges which meet at a vertex of the graph.

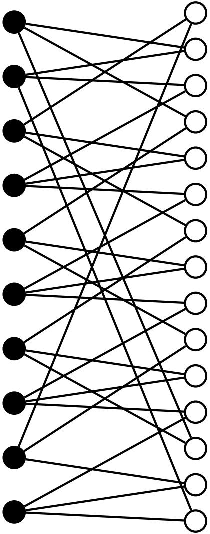
For example, $\{1, 2, 3\}$, $\{3, 5, 12\}$.



In fact since $nd = mr$, 15 is the best we can do.

Incidence graphs and duality

Given a hypergraph H, its *incidence graph* I has vertex set $V(H)$ (black vertices) and $E(H)$ (white vertices). There is an edge from v to e in I if and only if $v \in e$ in H.



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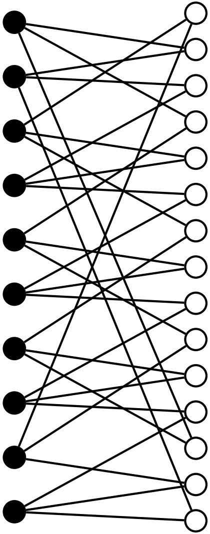
If H is a d-regular, r-uniform hypergraph, then I is a (d, r)-biregular bicoloured graph.



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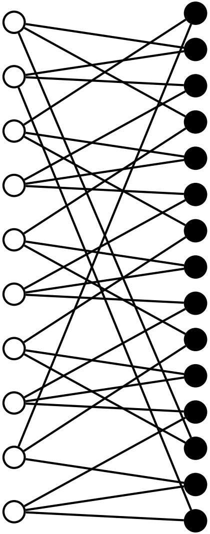
By swapping the colour classes in I we get the incidence graph I^* of the *dual* hypergraph H^*. This is r-regular and d-uniform.



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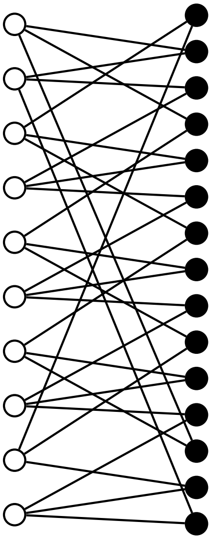
This is exactly what the previous construction was doing.



Incidence graphs and duality

Recall that in a Berge cycle, *all the hyperedges are distinct*.

Thus a cycle of length k in H corresponds to a cycle of length $2k$ in the incidence graph I.

And thus a cycle of length $2k$ in I^*.

And thus a cycle of length k in H^*.

So:

Observation

$\mathrm{girth}(H) = \mathrm{girth}(H^*)$.

This simple observation has some interesting consequences.

The Moore bound revisited

Let H be a d-regular, r-uniform hypergraph of girth $2k+1$.

$$|V(H)| \geq 1 + d(r-1)\frac{(d-1)^k(r-1)^k - 1}{(d-1)(r-1)-1} \tag{1}$$

Since $\mathrm{girth}(H^*) = 2k+1$, we also have:

$$|V(H^*)| \geq 1 + r(d-1)\frac{(d-1)^k(r-1)^k - 1}{(d-1)(r-1)-1}$$

$$|E(H^*)| \geq \frac{r}{d}\left(1 + r(d-1)\frac{(d-1)^k(r-1)^k - 1}{(d-1)(r-1)-1}\right)$$

$$|V(H)| \geq \frac{r}{d}\left(1 + r(d-1)\frac{(d-1)^k(r-1)^k - 1}{(d-1)(r-1)-1}\right) \tag{2}$$

If $r > d$, then (2) $>$ (1).

The Moore bound at girth 5

$d \setminus r$	2	3	4	5	6	7	8
2	5	13	25	41	61	85	113
3	10	31	64	109	166	235	316
4	17	57	121	209	321	457	617
5	26	91	196	341	526	751	1016
6	37	133	289	505	781	1117	1513
7	50	183	400	701	1086	1555	2108
8	65	241	529	929	1441	2065	2801

Table 2: Order bound for d-regular, r-uniform hypergraphs

$d \setminus r$	2	3	4	5	6	7	8
2	5	15	34	65	111	175	260
3	10	31	76	152	266	427	643
4	17	57	121	245	434	700	1058
5	26	91	196	341	606	982	1487
6	37	133	289	505	781	1267	1922
7	50	183	400	701	1086	1555	2360
8	65	241	529	929	1441	2065	2801

Table 3: Order bound taking account of duality

Cubic graphs and 3-regular, 3-uniform hypergraphs

A 3-regular, 3-uniform hypergraph of girth g has an incidence graph which is a cubic bipartite graph of girth $2g$. So the smallest hypergraphs can be determined from the list of smallest known cubic graphs of even girth.

$2g$	Graph
6	14
8	30
10	70
12	126
14	384
16	960
18	2,560

$2g$	Graph
20	5,376
22	16,206
24	49,608
26	109,200
28	415,104
30	1,143,408
32	3,650,304

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$2g$	Graph	Hypergraph
6	14	7
8	30	15
10	70	35
12	126	63
14	384	192
16	960	480
18	2,560	1,280

$2g$	Graph	Hypergraph
20	5,376	2,688
22	16,206	8,103
24	49,608	24,804
26	109,200	54,600
28	415,104	207,552
30	1,143,408	571,704
32	3,650,304	1,825,152

Cubic graphs and 2-regular, 3-uniform hypergraphs

A cubic graph has the same girth as its dual, which is a 2-regular, 3-uniform hypergraph.

g	Graph
3	4
4	6
5	10
6	14
7	24
8	30
9	58
10	70
11	112
12	126
13	272
14	384
15	620
16	960
17	2,176

g	Graph
18	2,560
19	4,324
20	5,376
21	16,028
22	16,206
23	49,326
24	49,608
25	108,906
26	109,200
27	285,852
28	415,104
29	1,141,484
30	1,143,408
31	3,649,794
32	3,650,304

Cubic graphs and 2-regular, 3-uniform hypergraphs

A cubic graph has the same girth as its dual, which is a 2-regular, 3-uniform hypergraph.

g	Graph	Hypergraph
3	4	6
4	6	9
5	10	15
6	14	21
7	24	36
8	30	45
9	58	87
10	70	105
11	112	168
12	126	189
13	272	408
14	384	576
15	620	930
16	960	1,440
17	2,176	3,264

g	Graph	Hypergraph
18	2,560	3,840
19	4,324	6,486
20	5,376	8,064
21	16,028	24,042
22	16,206	24,309
23	49,326	73,989
24	49,608	74,412
25	108,906	163,359
26	109,200	163,800
27	285,852	428,778
28	415,104	622,656
29	1,141,484	1,712,226
30	1,143,408	1,715,112
31	3,649,794	5,474,691
32	3,650,304	5,475,456

Cubic graphs from hypergraphs

We can find the best 2-regular, 3-uniform and 3-regular, 3-uniform hypergraphs by looking at the list of the best cubic graphs.

Can we go the other way? Can we find 2 or 3-regular, 3-uniform hypergraphs of large girth whose duals or incidence graphs will be new best cubic graphs of given girth?

We seek a method of construction of “interesting” hypergraphs with given parameters.

There are hypergraph analogues of Cayley graphs.

Cayley graphs

Given a (finite) group G and an inverse-closed subset S of G:

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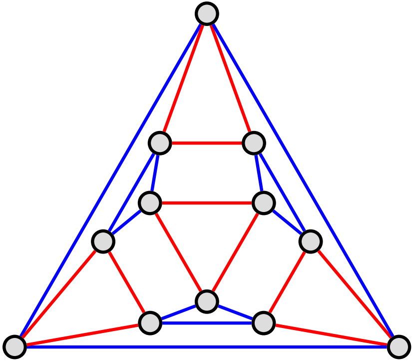
$\mathrm{Cay}(G, S)$ has vertex set G and edge set $\{\{g, gs\} : g \in G, s \in S\}$.



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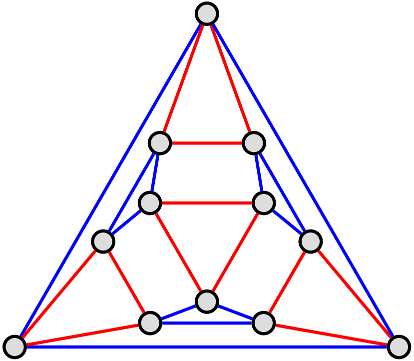
The graph has order $|G|$ and degree $|S|$.



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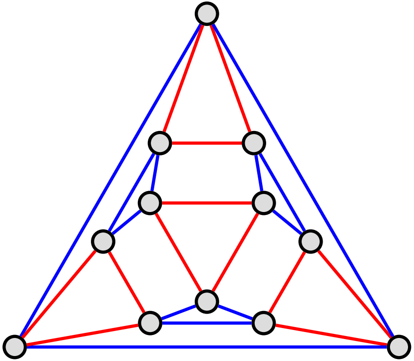

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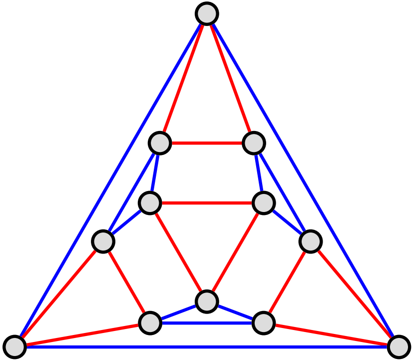

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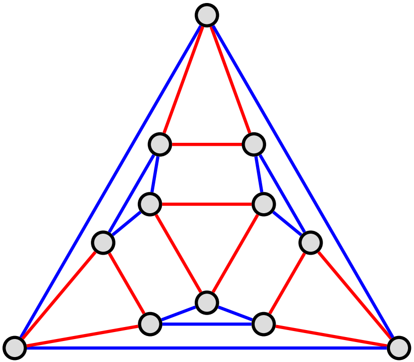
These graphs are highly symmetric, since (left) multiplication of all vertices by any element of the group induces a graph automorphism.



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Cayley hypergraphs

M. Buratti, *Cayley, Marty and Schreier Hypergraphs*, 1994.

Let G be a finite group, $S \subseteq G \setminus \{1\}$ and let $t \geq 2$. The *t-Cayley hypergraph* $t\text{-Cay}(G, S)$ has vertex set G and hyperedge set

$$\{\{g, gs, \ldots, gs^{t-1}\} : g \in G, s \in S\}$$

Note that if $t = 2$, then $2\text{-Cay}(G, S) = \text{Cay}(G, S \cup S^{-1})$.

This definition is easy to work with but has some caveats.

- If $t > \min\{o(s) : s \in S\}$ then $t\text{-Cay}(G, S)$ is not uniform.
- If $2 < t < \max\{o(s) : s \in S\}$ then $t\text{-Cay}(G, S)$ is not linear.
- G acts by left multiplication as a regular group of automorphisms.

To get a d-regular, r-uniform linear hypergraph we want a set S of d (independent) elements of order exactly r.

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To get a d-regular, r-uniform linear hypergraph we want a set S of d (independent) elements of order exactly r. The hyperedges of $r\text{-Cay}(G, S)$ are the left cosets of $\langle s \rangle$ for all $s \in S$.

Other definitions of Cayley hypergraph are possible, but this is the most useful one for our needs.

Finding good 3-Cayley hypergraphs — candidate groups

The dual of a 2-regular 3-Cayley hypergraph is **a cubic graph of the same girth**. The basic idea is to find a 2-regular 3-Cayley hypergraph of large girth.

To create a 2-regular, 3-uniform 3-Cayley hypergraph we want a group of the form $G = \langle a, b \,|\, a^3, b^3, \ldots \rangle$. What are these groups and how do we find them?

- G must contain at least two distinct subgroups of order 3.
- G cannot have an index 2 subgroup. So if G has even order:
  - $|G| \equiv 0 \pmod 4$;
  - G is not nilpotent;
  - If $|G| = 3 \times 2^k$ for some $k \geq 1$, then the Sylow 2-subgroup of G is normal.

These restrictions allow us to identify all such groups up to order 1000 and most up to 2000. Other good candidate groups are the perfect groups; these include the simple groups like $PSL(2, q)$ which are often useful.

The range of orders of interest goes up to about 2M. We find as many groups as we can, including groups generated by two random elements of a suitable symmetric group. Then take all possible direct products, provided the resulting group is still (3,3)-generated.

Construction method

- Pick one of the 34,970 candidate (3,3)-generated groups G.
- Using `GAP`, find orbit representatives of pairs a, b of elements of order 3 generating G. (Or a random sample if there are too many.)
- Compute the girth of the hypergraph $H = 3\text{-Cay}(G, \{a, b\})$.

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- Compute the girth of the hypergraph $H = 3\text{-Cay}(G, \{a, b\})$. This is the smallest g such that there exists a word $\alpha_1\beta_2\alpha_3\beta_4 \cdots \alpha_{g-1}\beta_g = 1$, where each $\alpha_i \in \{a, a^{-1}\}$ and $\beta_j \in \{b, b^{-1}\}$.
- H has $|G|$ vertices and $\frac{2}{3}|G|$ hyperedges.
- The dual H^* is a cubic bipartite graph of order $\frac{2}{3}|G|$ and also has girth g.
- A way to view the cubic graph H^* is as a bipartite graph with partitions the left cosets of $\langle a \rangle$ and $\langle b \rangle$, with an edge from $x\langle a \rangle$ to $y\langle b \rangle$ whenever $x\langle a \rangle \cap y\langle b \rangle \neq \emptyset$.

Results

g	Current	Our best	Group
4	6	6	$\mathbb{Z}_3 \times \mathbb{Z}_3$
6	14	14	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$
8	30	40	$PSL(2,5)$
10	70	112	$\mathbb{Z}_2^3 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$
12	126	162	$(\mathbb{Z}_3 \times (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_3)) \rtimes \mathbb{Z}_3$
14	384	624	$(\mathbb{Z}_{13} \rtimes \mathbb{Z}_3) \times SL(2,3)$
16	960	1,008	$\mathbb{Z}_3^2 \times PSL(2,7)$
18	2,560	2,688	$A_4 \times SL(2,7)$
20	5,376	12,096	$\mathbb{Z}_3 \times A_4 \times PSL(2,8)$
22	16,206	23,328	$((\mathbb{Z}_3^3 \rtimes \mathbb{Z}_2^2) \times (\mathbb{Z}_2^2 \rtimes (\mathbb{Z}_9 \rtimes \mathbb{Z}_3))) \rtimes \mathbb{Z}_3$
24	49,608	35,640	$(\mathbb{Z}_3^3 \rtimes \mathbb{Z}_3) \times PSL(2,11)$
26	109,200	109,200	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times PSL(2,25)$
28	415,104	368,640	$(\mathbb{Z}_2^4 \rtimes SL(2,5)) \times SL(2,3) \times A_4$
30	1,143,408	806,736	$(\mathbb{Z}_7^3 \cdot PSL(3,2)) \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$
32	3,650,304	1,441,440	$\mathbb{Z}_3 \times PSL(2,11) \times PSL(2,13)$

Table 4: Smallest cubic graphs of even girth g

Excision

Recall that our construction yields a bipartite graph with partitions the left cosets of $\langle a \rangle$ and $\langle b \rangle$, with an edge from $x\langle a \rangle$ to $y\langle b \rangle$ whenever $x\langle a \rangle \cap y\langle b \rangle \neq \emptyset$.

So this will only ever yield a graph of even girth. What about odd girths?

Theorem (N. Biggs, 1998)

Let Γ be a cubic graph of girth $g \geq 4$ and order n, and let $r = \lfloor \frac{g}{4} \rfloor$. Then there exists a cubic graph Γ' of order $n - \epsilon$ and girth $g - 1$, where

$$\epsilon = \begin{cases} 2^{r+1} - 2, & g \equiv 0, 1 \pmod 4; \\ 3 \times 2^r - 2, & g \equiv 2, 3 \pmod 4. \end{cases}$$

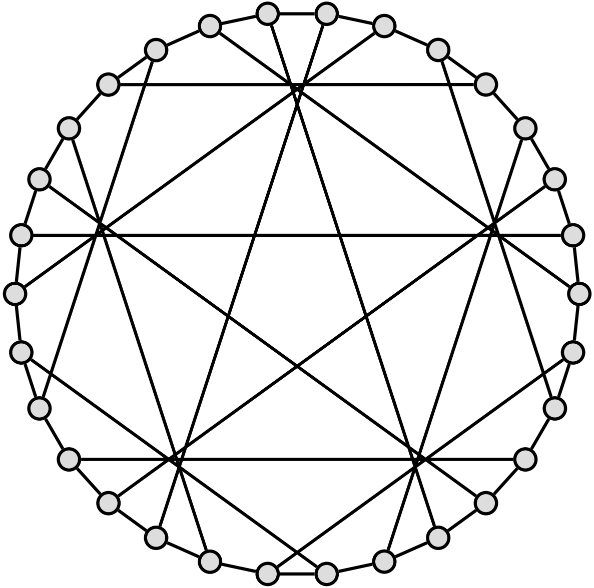
Idea: *Excise a tree of depth $r - 1$ from an edge, or a tree of depth r rooted at a vertex. Then join back up the vertices of valency 2 thus created.*

Observation: *This gives a lower bound for the number of vertices which can be excised. It is frequently possible to do better by carrying out the initial excision then attempting to chop out smaller trees from the resulting graph.*

Excision example

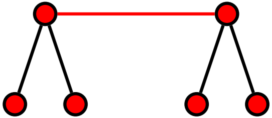
Begin with a cubic graph of girth 8 with order 30.

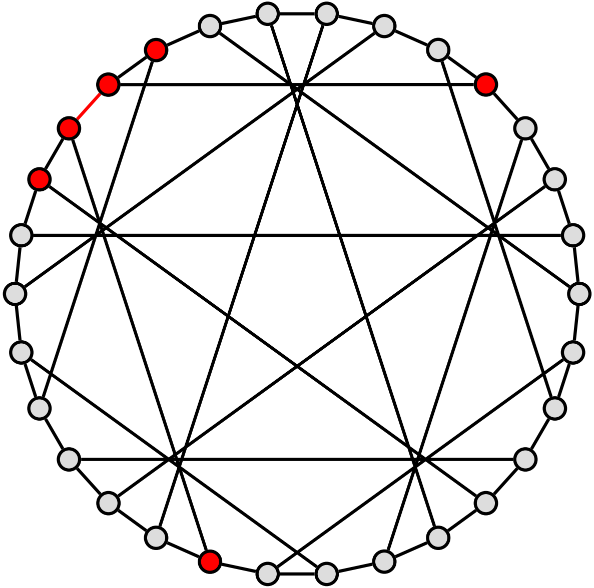
This is the *Tutte-Coxeter graph* or *Tutte 8-cage*.



Excision example

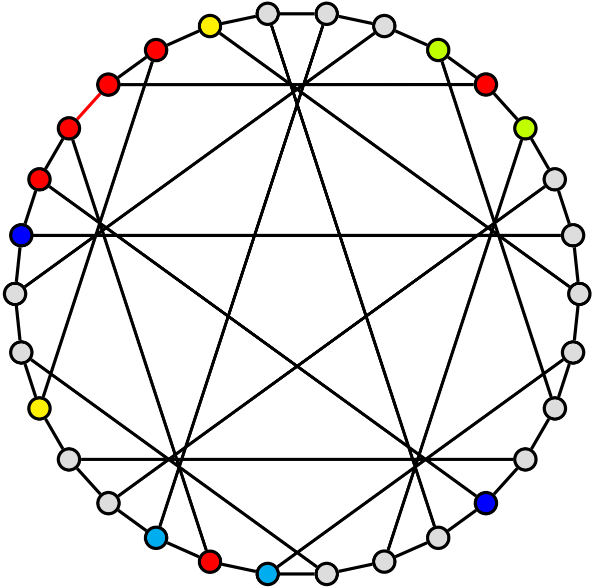
Choose an edge and mark a tree of depth 1 from that edge.





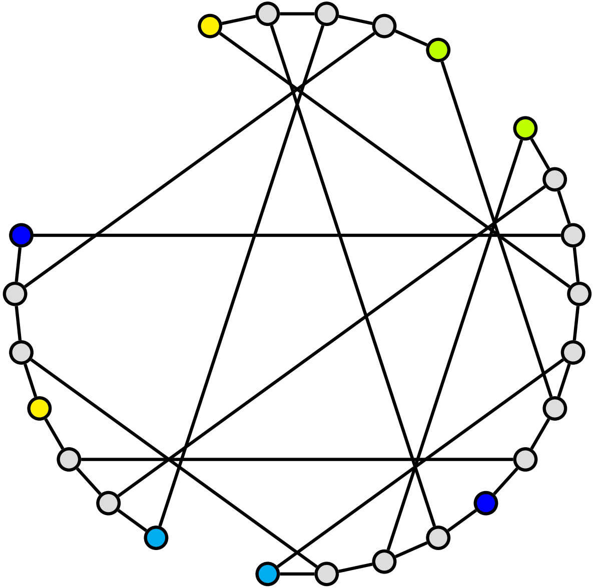
Excision example

Identify the two other neighbours of all the leaf nodes in the tree to be excised.



Excision example

Remove the vertices in the tree.

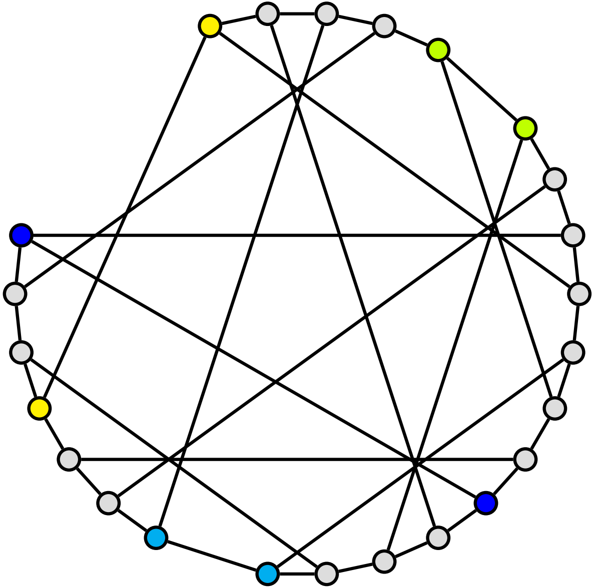


Excision example

Finally, join up the identified vertices of valency 2. These were at distance 2 in the original graph.

The result is a cubic graph of order 24 and girth 7.

In fact this is the McGee graph which is the unique cage of girth 7.



New table

g	Graph	Description	g	Graph	Description
3	4	K_4	18	$2,560$	Exoo
4	6	$K_{3,3}$	19	$4,324$	Hoare
5	10	Petersen	20	$5,376$	Exoo
6	14	Heawood	21	$16,028$	Exoo
7	24	McGee	22	$16,206$	Biggs/Hoare
8	30	Tutte	23	$35,446$	* NEW *
9	58	Brinkmann/McKay/Saager	24	$35,640$	* NEW *
10	70	O'Keefe/Wong	25	$108,906$	Exoo
11	112	McKay/Myrvold; Balaban	26	$109,200$	Bray/Parker/Rowley
12	126	Benson	27	$285,852$	Bray/Parker/Rowley
13	272	McKay/Myrvold; Hoare	28	$368,640$	* NEW *
14	384	McKay; Exoo	29	$805,746$	* NEW *
15	620	Biggs	30	$806,736$	* NEW *
16	960	Exoo	31	$1,440,338$	* NEW *
17	$2,176$	Exoo	32	$1,441,440$	* NEW *

Thoughts and next steps

The graphs we construct are edge-transitive but not necessarily vertex-transitive. So the space is less thoroughly searched by previous authors.

The graphs of Bray/Parker/Rowley are coset graphs, as are our constructions. Is the graph we found at 26 isomorphic to the known graph of the same order?

A d-regular, r-uniform hypergraph is essentially a (d, r)-biregular bipartite graph. Are there any results from the investigations of these graphs which might be useful?

Is there a way to generate non-bipartite graphs?

Are there other families of groups in the range of interest which are worth a look?

Reference: G. Erskine and J. Tuite. *Small graphs and hypergraphs of given degree and girth.* `https://arxiv.org/abs/2201.07117`.