On Mixed Graphs, Girth, Diameter and Geodecity

Geoffrey Exoo

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Outline

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1. Mixed cages and strongly regular graphs

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2. Directed geodetic cages

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3. Mixed degree/diameter graphs

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3. Mixed degree/diameter graphs
4. Comments on exhaustive searches
5. Bipartite mixed degree/diameter graphs

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- ★ *ideg(v)*: the in-degree of v is the number of arcs (w, v).

Definition

A mixed graph is regular if $deg(v)$ and $odeg(v)$ are constant as v ranges over $V(G)$, in which case we denote the degree by r and the out-degree by z. If, in addition, $ideg(v)$ is constant the graph is totally regular.

Definition

The girth of a mixed graph is the length of a shortest cycle.

Definition

The diameter of a mixed graph is the maximum length of a shortest path from any vertex u to any vertex v.

Definition

The geodecity of a mixed graph is the maximum k such that there is most one walk of length at most k from vertex u to vertex v.

Definition

An (r, z, g)-graph is a mixed graph with degree r, out-degree z, and girth g.

Definition

A mixed (r, z, g)-cage is an (r, z, g)-graph of minimum possible order. Let $f(r, z, g)$ denote the order of a mixed (r, z, g)-cage.

Mixed Cages

Moore Bound

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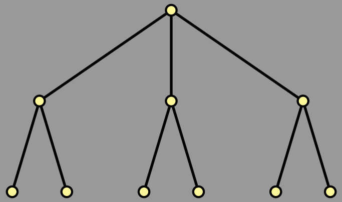
A Moore Tree of Degree 3 and Radius 2



Bounds for Mixed Cages

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$$f(r,1,g) \geq \sum_{i=0}^{g-1} n(r, \mathtt{min}(i, g-i-1))$$

From *Mixed Cages*, Gabriela Araujo-Pardo, César Hernández-Cruz, J.J. Montellano-Ballesteros, Graphs and Combinatorics, 35 (2019).

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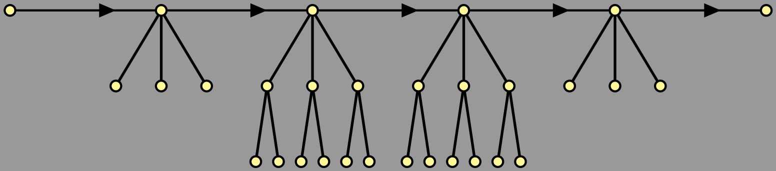
Example: AHM Bound for $r = 3$, $z = 1$ and $g = 6$



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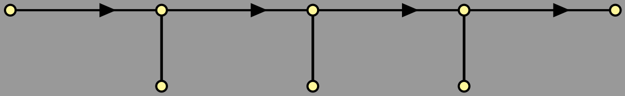

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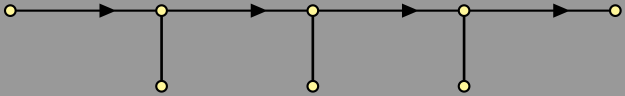

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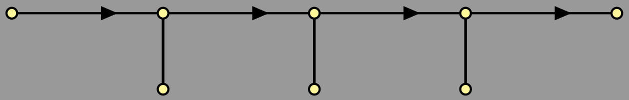

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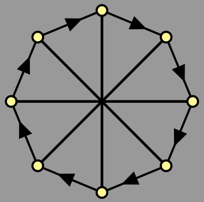

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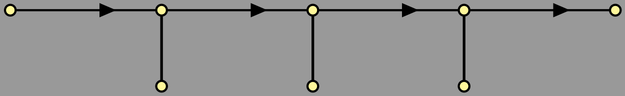

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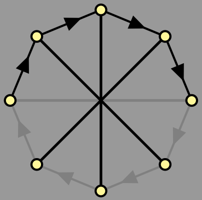

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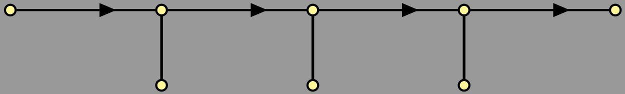

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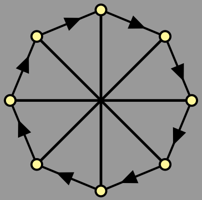

Known AHM Graphs

Theorem

$f(1, 1, g) = 2g - 2$

$f(2, 1, g) = \lceil g^2/2 \rceil$ *[AHM]*

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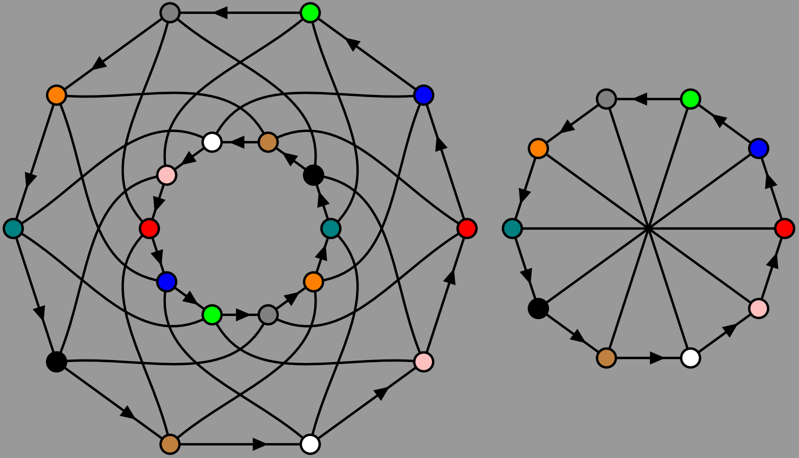

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  Every pair of adjacent vertices has λ common neighbors.
  Every pair of non-adjacent vertices has μ common neighbors.

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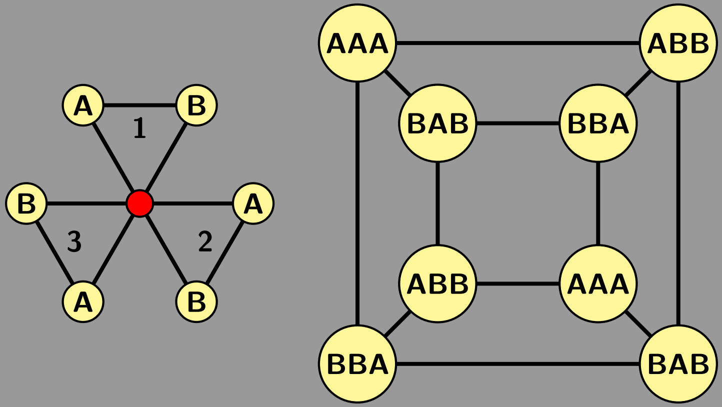
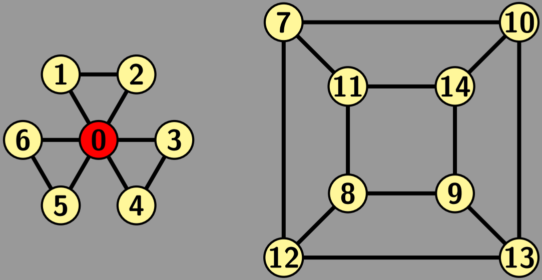


Figure: The complement of the line graph of K_6 or the collinearity graph of the generalized quadrangle $GQ(2, 2)$.

A Spanning Tree for $SRG(15, 6, 1, 3)$



The spanning tree consists of the following edges:

The 6 edges incident with vertex 0.

The 8 edges from vertices 1 and 2 to the cube.

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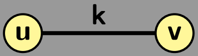

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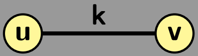

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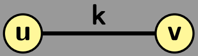

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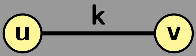

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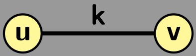

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- If $u < v$ then vertex i in the $\vec{C}_6$ at vertex u is adjacent to vertex $i + k$ mod 6 in the $\vec{C}_6$ at vertex u.

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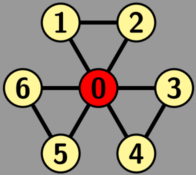

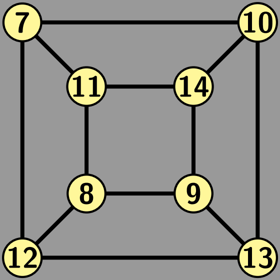

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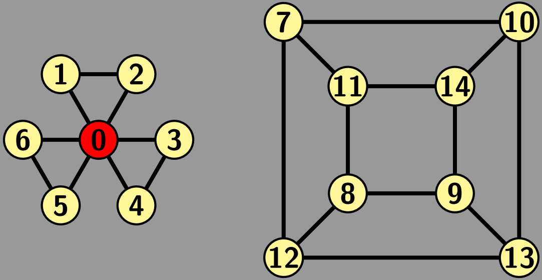

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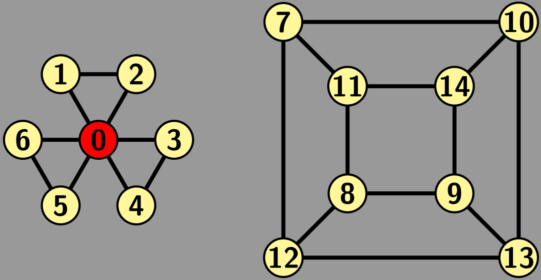

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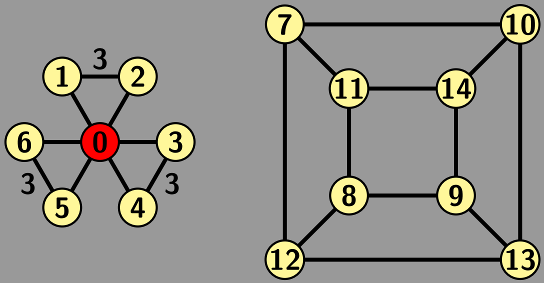

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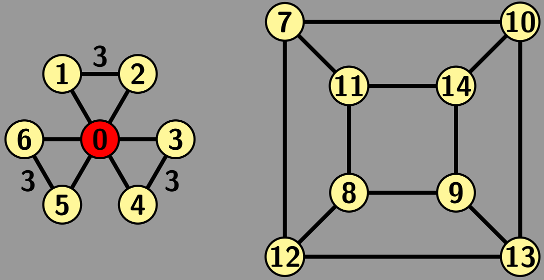

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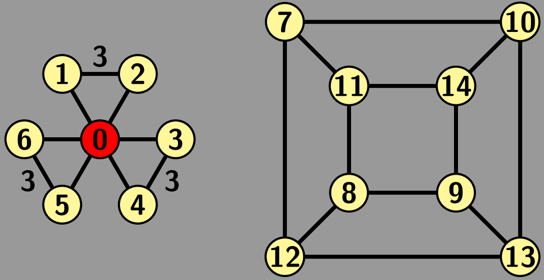

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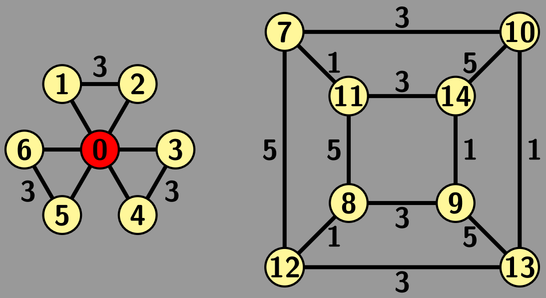

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Voltages among edges in the cube are all odd.

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Directed Geodetic Cages

Digraph Geodecity

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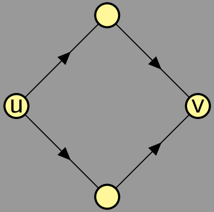
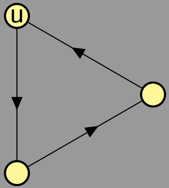
Examples from Grahame's Talk

Not 2-geodetic



Not 3-geodetic



Geodetic Cages

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Questions

- Find the smallest k-geodetic digraph with out degree d?

Geodetic Cages

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Notation

- In $\vec{K}_{m,n}$ all arcs are directed from the m-set to the n-set.

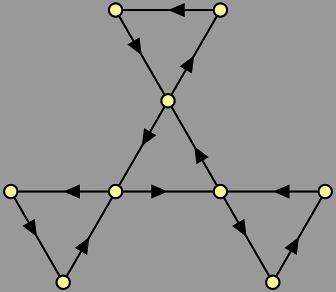
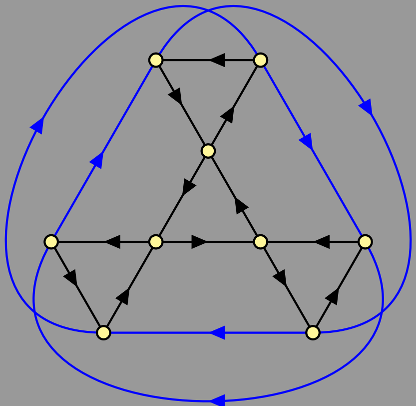


Figure: The two $(2,2)$-geodetic cages (due to Erskine and Tuite).



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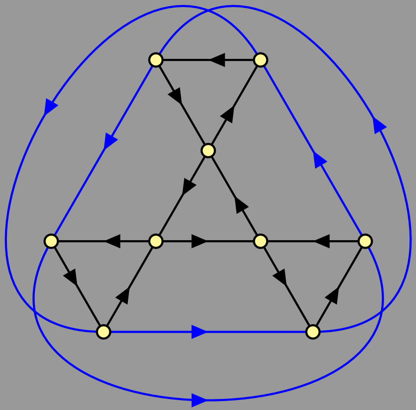

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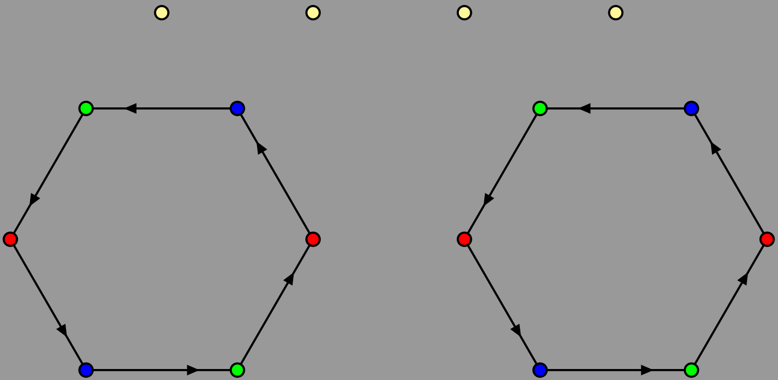

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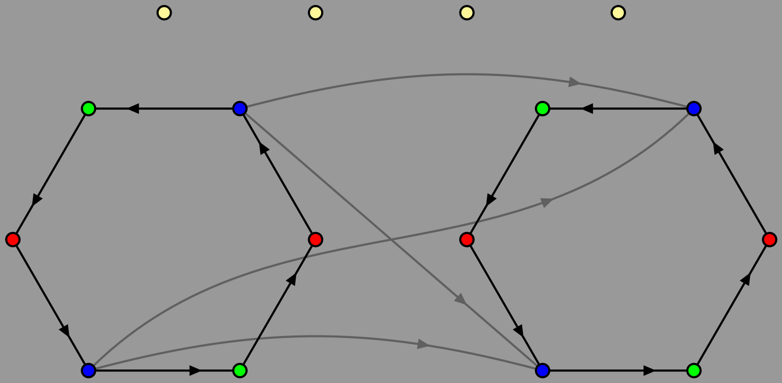

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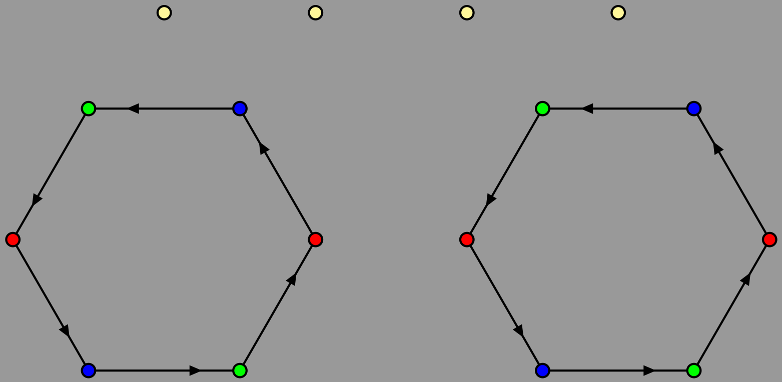

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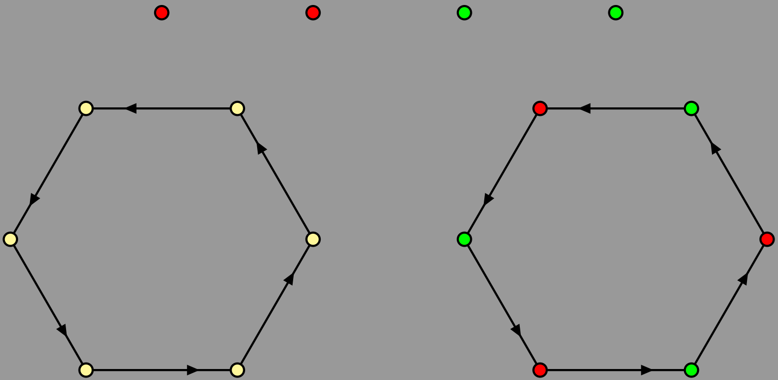

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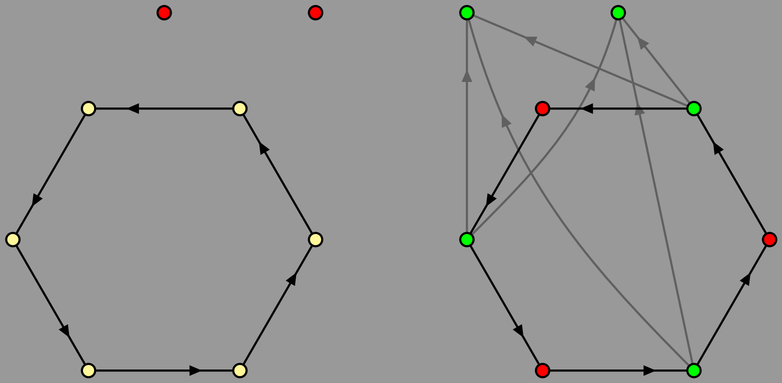

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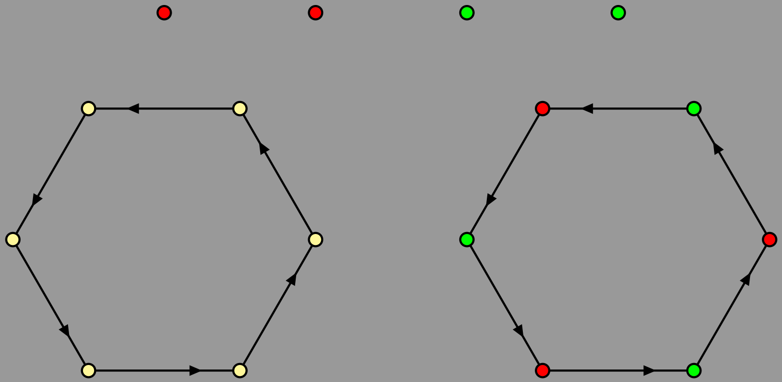

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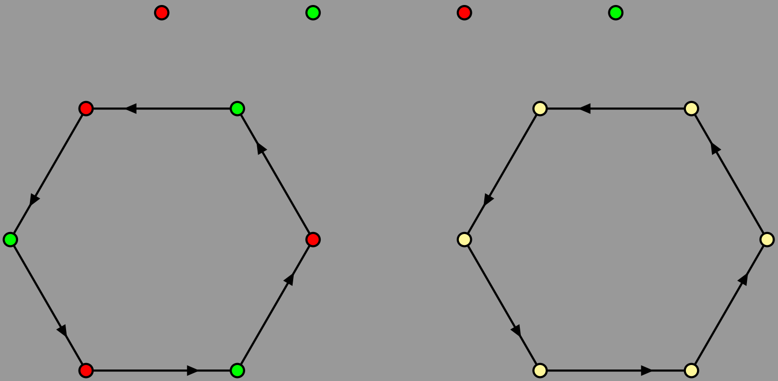

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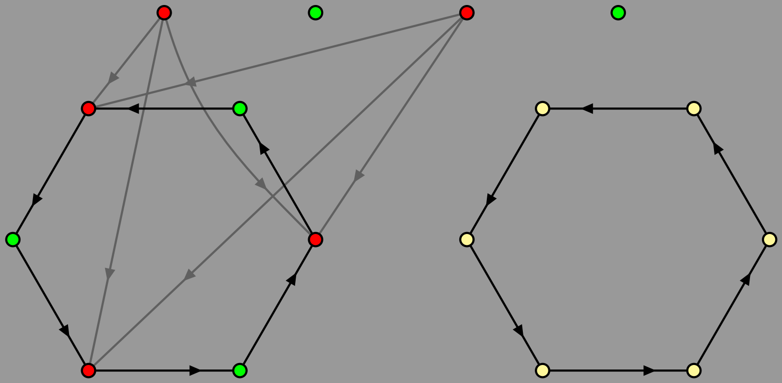

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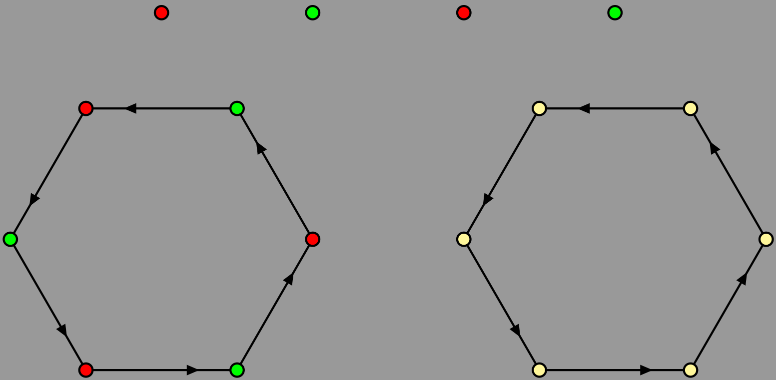

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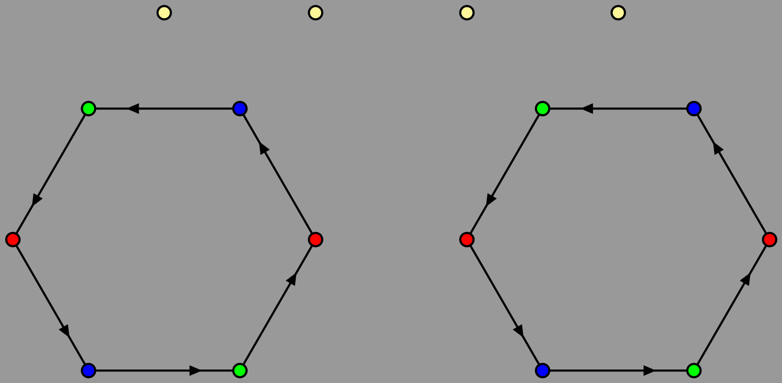

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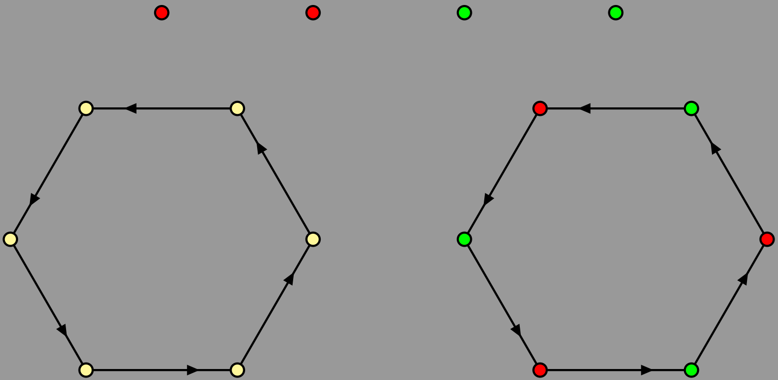

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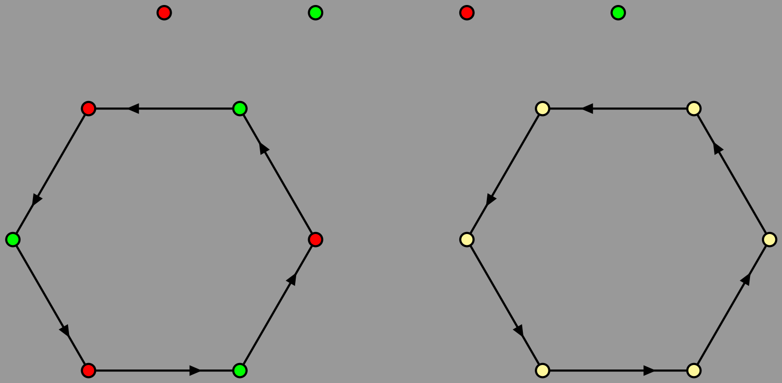

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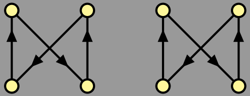

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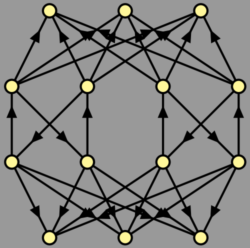

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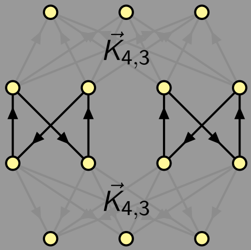

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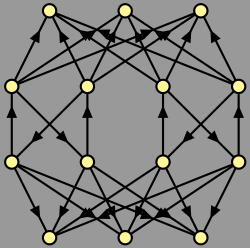

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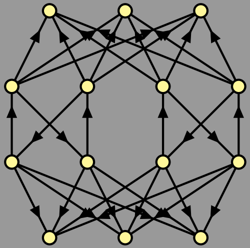

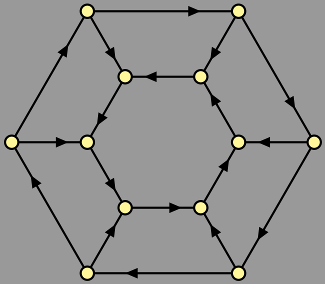

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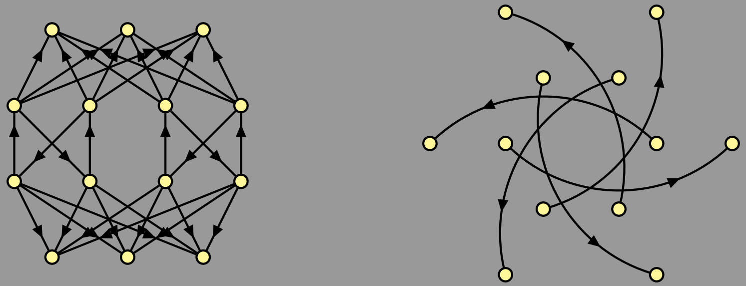

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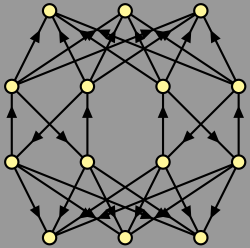

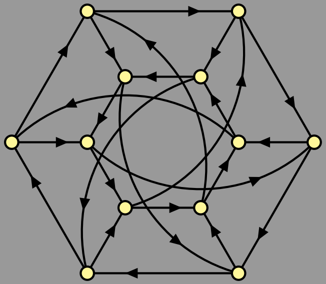

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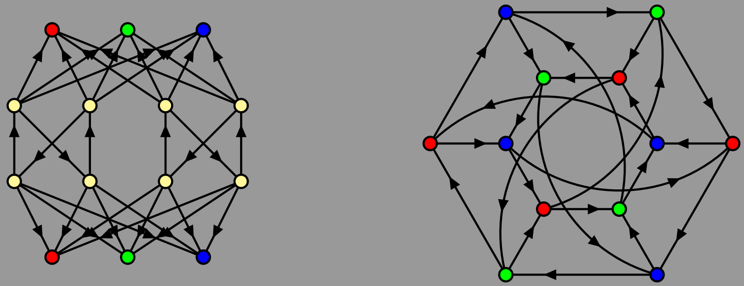

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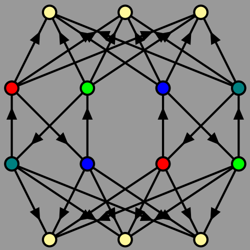

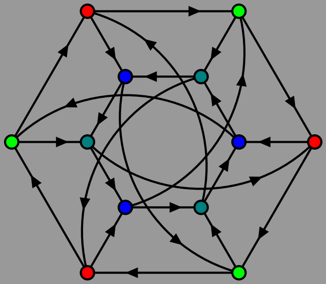

Another Table

Degree	Geodecity	Moore Bound	Best Graph	Excess
2	2	7	9	2
3	2	13	16	3
4	2	21	26	5
5	2	31	39	8
6	2	43	52	9
7	2	57	68	11
8	2	73	87	14

Figure: Exact values are in red. Entries in blue correspond to graphs that are not in-regular. Entries in green are exact for totally regular graphs.

Mixed Graphs
Degree/Diameter Problem

Joint work with:

C. Dalfó

G. Erskine

M. A. Fiol

N. López

A. Messegué

J. Tuite

(r, z, k)-Graphs

- An (r, z, k)-graph is a mixed graph with:
  - undirected degree r
  - directed outdegree z
  - diameter k
- Problem: Find (r, z, k)-graphs of maximum order – this order denoted by $f(r, z, k)$.
- Focus on $f(1, 1, k)$.

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- Find an upper bound for $f(1,1,11)$?

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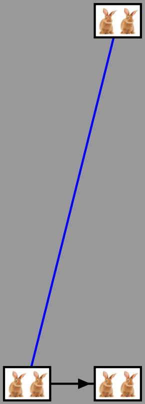

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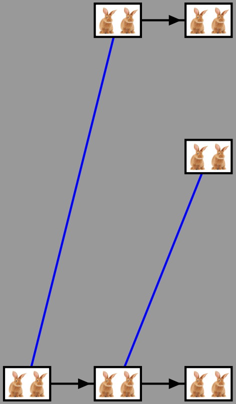

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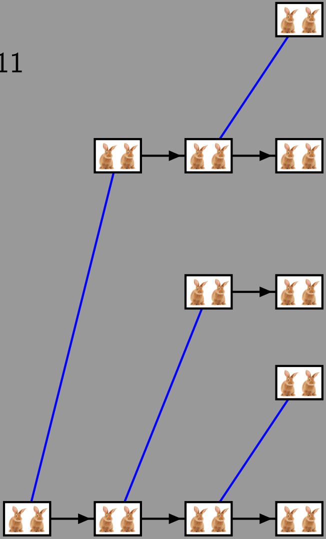
$f(1,1,3) \leq 11$



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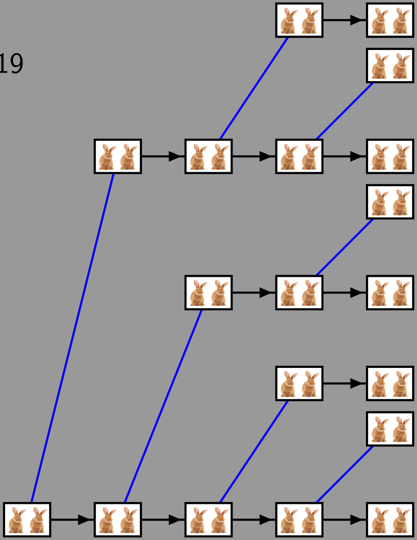
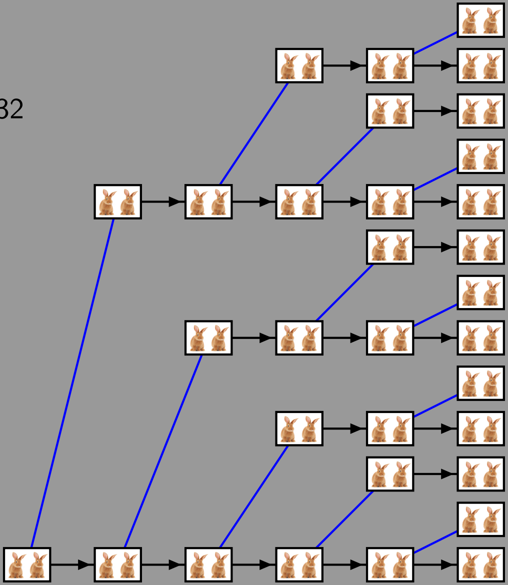
$f(1,1,4) \leq 19$



Fibonacci's Rabbits

$f(1,1,5) \leq 32$



Diameter k	Lower Bound	$f(1,1,k)$	Fibo Bound
2		6	6
3		10	11
4		14	19
5		?	32

Figure: Table of bounds for $f(1,1,k)$ and small k. Exact values for $k = 2,3,4$ due to Dalfó, Fiol, López.

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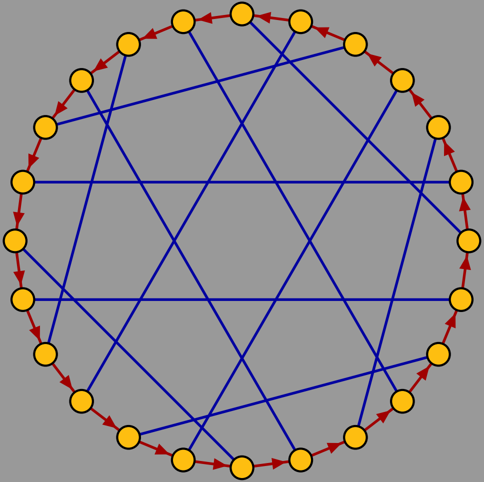
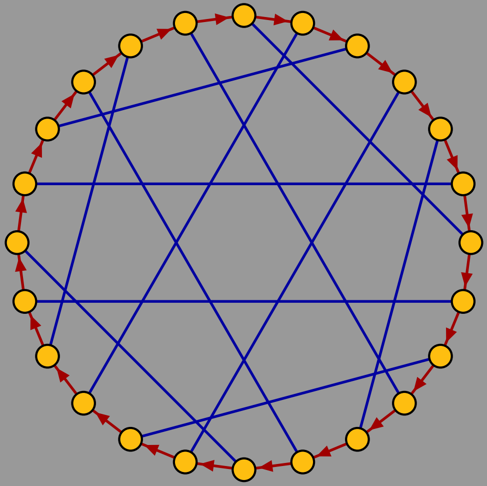


Figure: Two of the three $(1, 1, 5)$-graphs of order 24.



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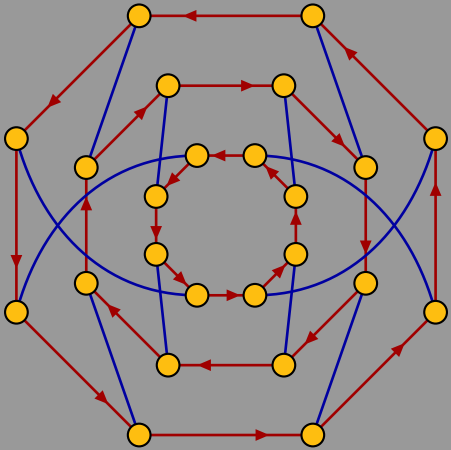

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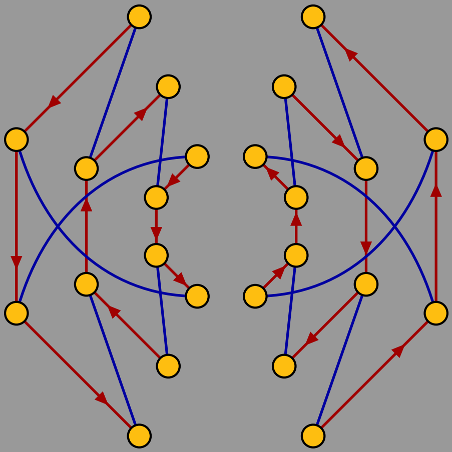

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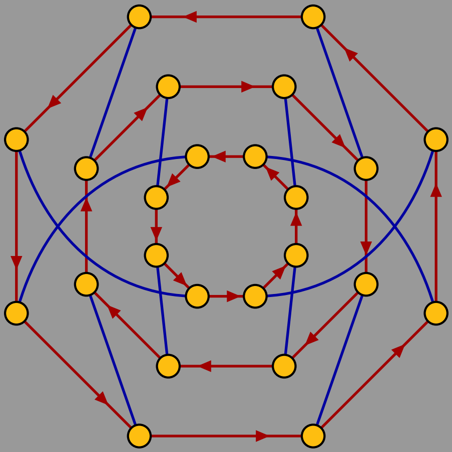

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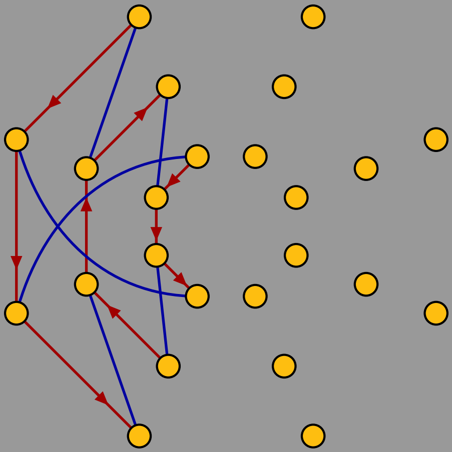

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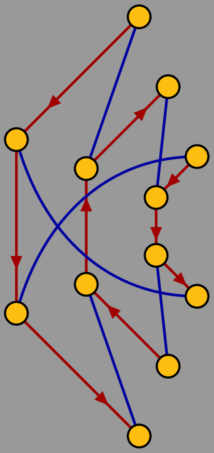

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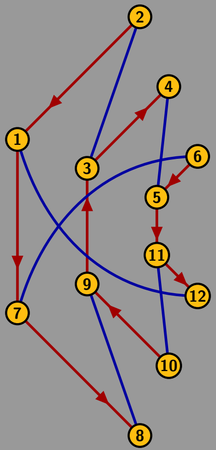

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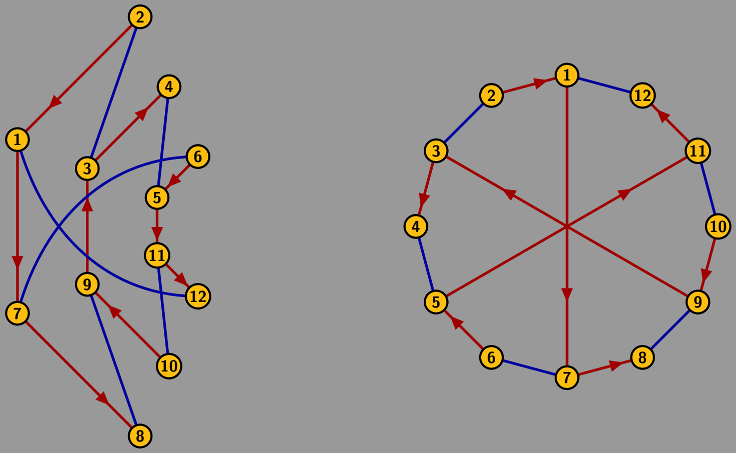

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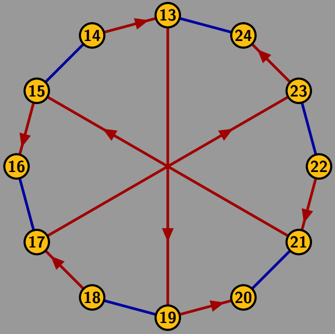

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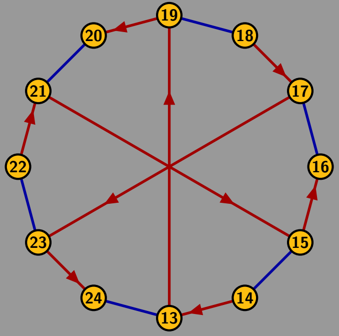

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A 3D view of the third $(1, 1, 5)$-graph of order 24.

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The underlying graph for any totally regular diameter 5 example of order 26 has girth 4.

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How to turn degree/diameter search into avoidance search?

Search for totally regular $(1, 1, 5)$-graphs of order 24

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- There are only 23 780 814 cubic graphs of order 24 with girth $g \geq 4$.

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- There are only 23 780 814 cubic graphs of order 24 with girth $g \geq 4$.
- For each graph, search for an orientation of diameter 5.

Search for totally regular $(1, 1, 5)$-graphs of order 24

- Consider the underlying cubic graph.
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- There are only 23 780 814 cubic graphs of order 24 with girth $g \geq 4$.
- For each graph, search for an orientation of diameter 5.
- This is an avoidance search, avoiding edge orientations that increase the diameter to more than 5.

Search for totally regular $(1, 1, 5)$-graphs of order 26

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- Need to eliminate girth 4 by other means.

Diameter k	Lower Bound	$f(1,1,k)$	Upper Bound
2		6	
3		10	
4		14	
5		24	
6	34		48

Figure: Table of bounds for $f(1,1,k)$ and small k.

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Figure: Table of bounds for $f(1,1,k)$ and small k.

Note: Miguel Pizaña and Claudia De la Cruz are working on methods that will handle these problems more easily.

Mixed Degree Diameter Graphs Bipartite

Joint work with:

D. Buset

C. Dalfó

G. Erskine

M. A. Fiol

T. Jajcayova

N. López

A. Messegué

J. Tuite

Largest bipartite $(1, 1, k)$-graphs

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- And showed that Moore graphs exist for the case $(1, 1, 3)$.

Largest bipartite $(1, 1, k)$-graphs

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- And showed Moore graphs do not exist for diameter $k > 3$.
- And showed that Moore graphs exist for the case $(1, 1, 3)$.
- And found the two largest $(1, 1, 3)$ graphs.

Yet Another Table

k	Moore Bound	Best Graph	Optimal?
3	8	8	Yes
4	14	12	Yes
5	24	18	Yes
6	40	30	Maybe

Figure: Bipartite $(1, 1, k)$-graphs.

$(1, 1, 3)$-Graphs

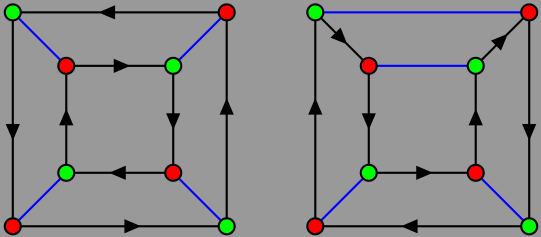


Figure: The Graphs of Dalfó, Fiol, and López.

$(1, 1, 4)$-Graphs

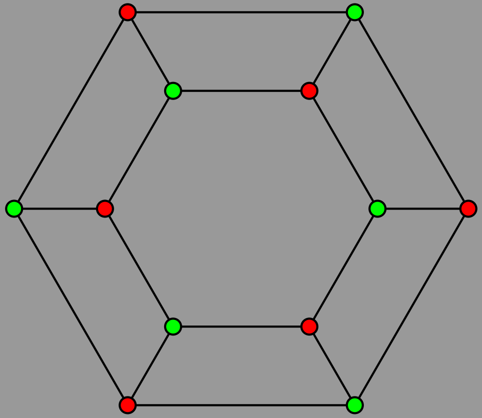


Figure: The Five Graphs of Erskine

$(1, 1, 4)$-Graphs

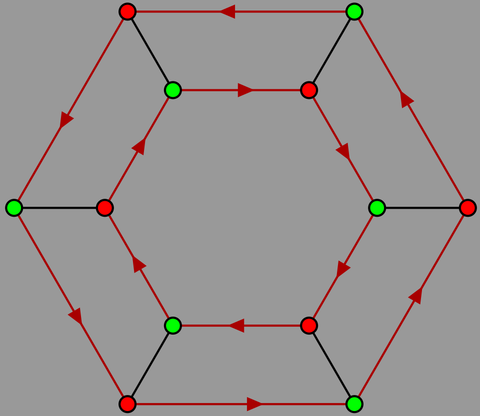


Figure: Two Directed 6-Cycles (version 1)

$(1,1,4)$-Graphs

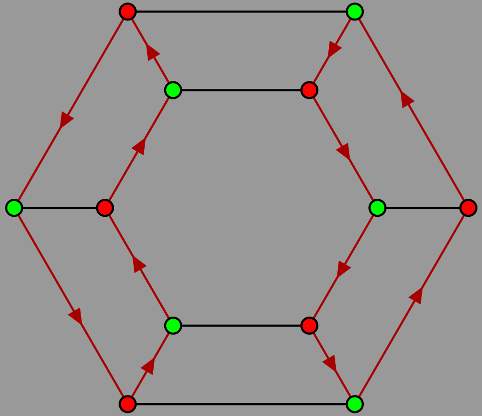


Figure: Two Directed 6-Cycles (version 2)

$(1, 1, 4)$-Graphs

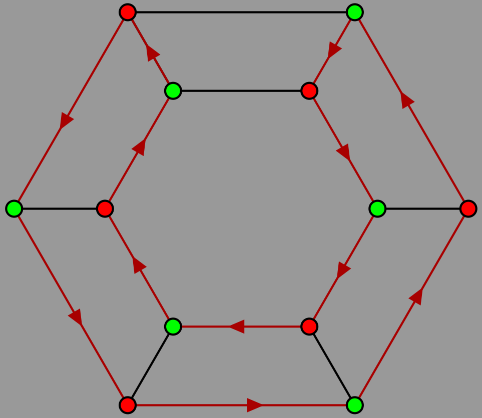


Figure: A Directed 12-Cycle

$(1, 1, 4)$-Graphs

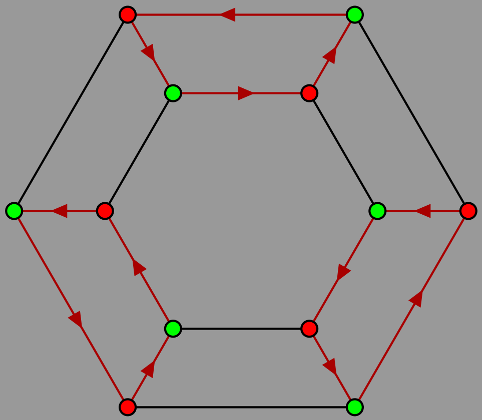


Figure: Three Directed 4-Cycles

$(1, 1, 4)$-Graphs

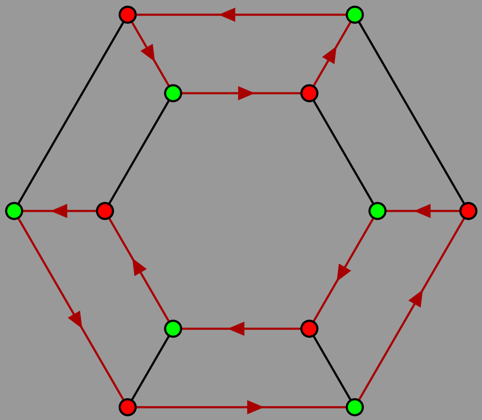


Figure: One Directed 4-Cycle and One Directed 8-Cycle

(1, 1, 5)-Graphs

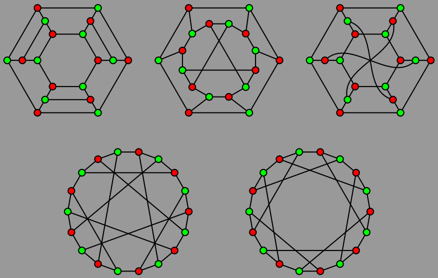
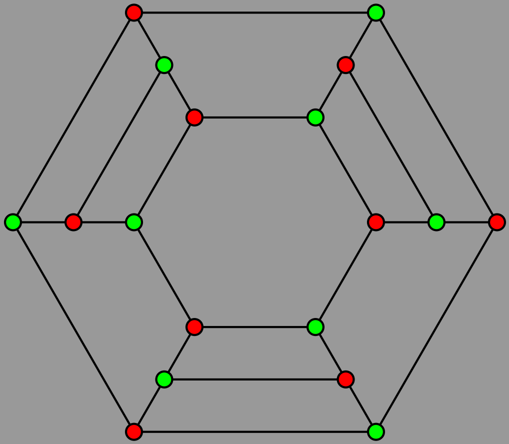
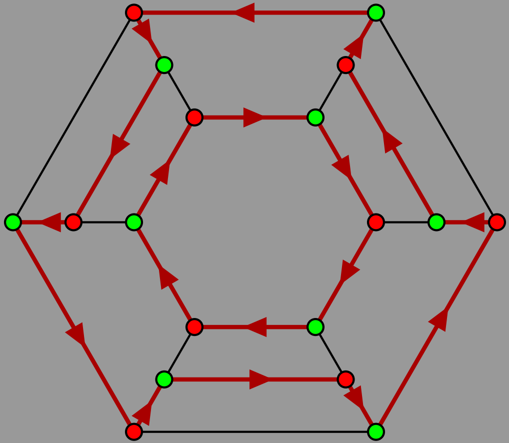


Figure: The five largest (order 18) biparitite cubic graphs with (1, 1, 5)-orientations, with a total of 24 (1, 1, 5) orientations.

$(1, 1, 5)$-Graphs



$(1, 1, 5)$-Graphs



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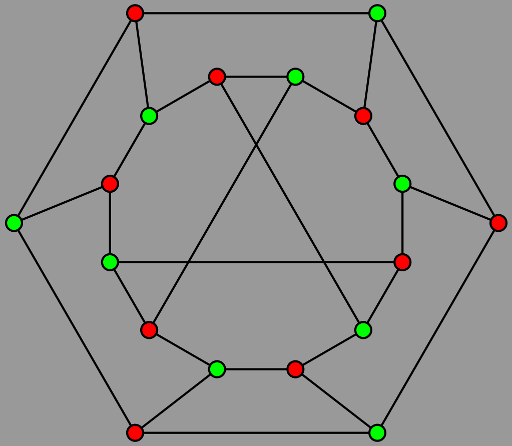
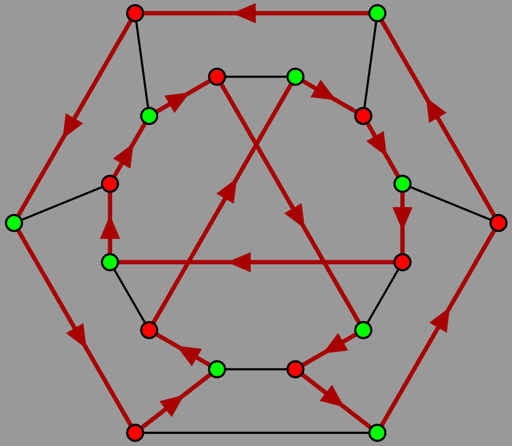


$(1, 1, 5)$-Graphs



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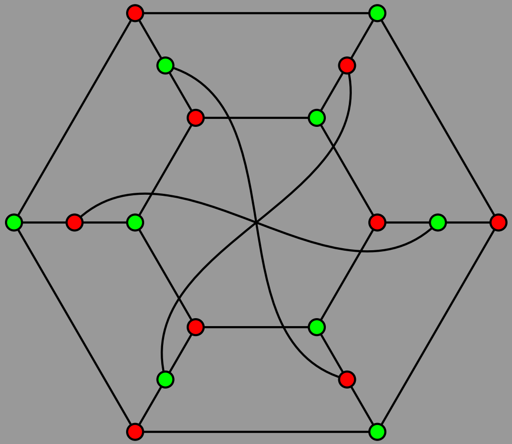

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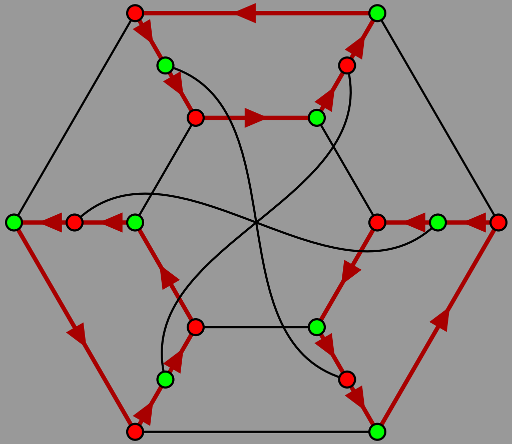

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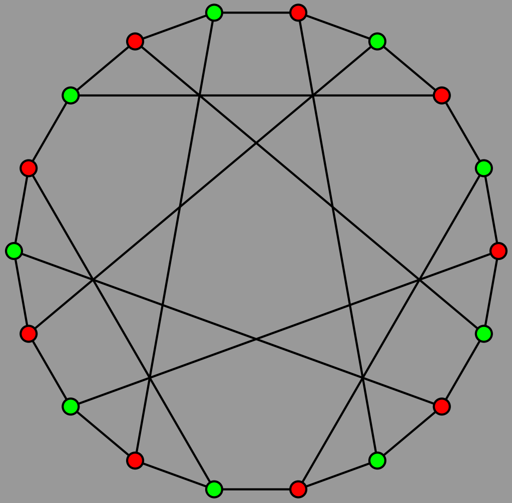

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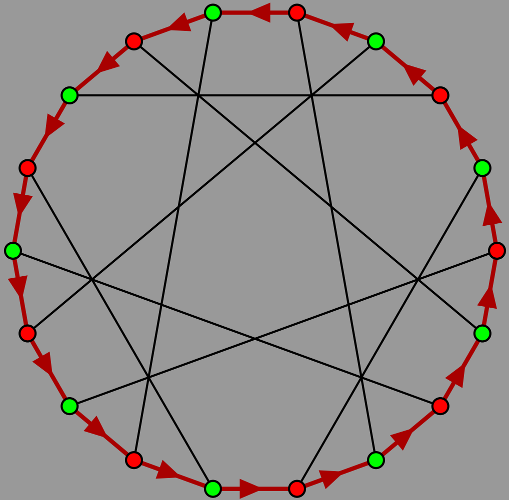

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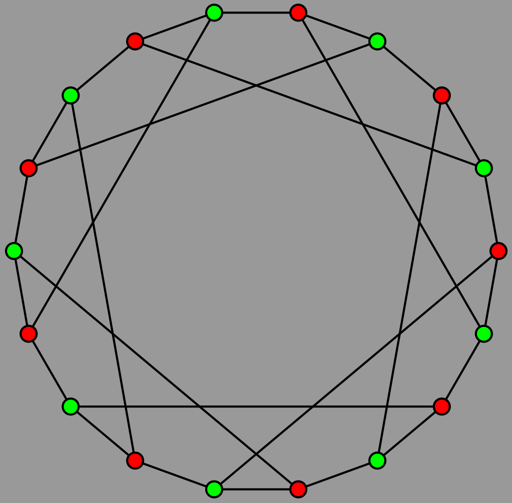

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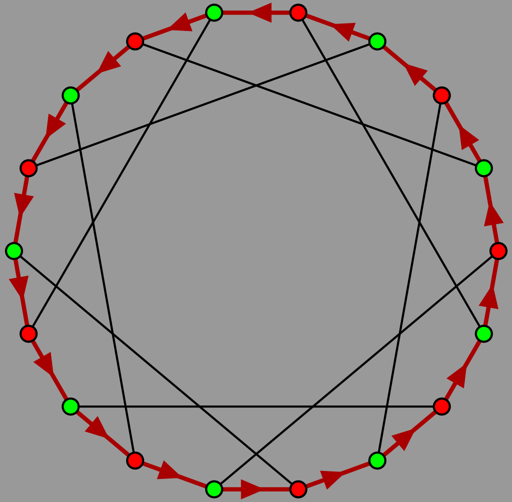

$(1, 1, 6)$ Totally Regular Bipartite Graph of Order 30

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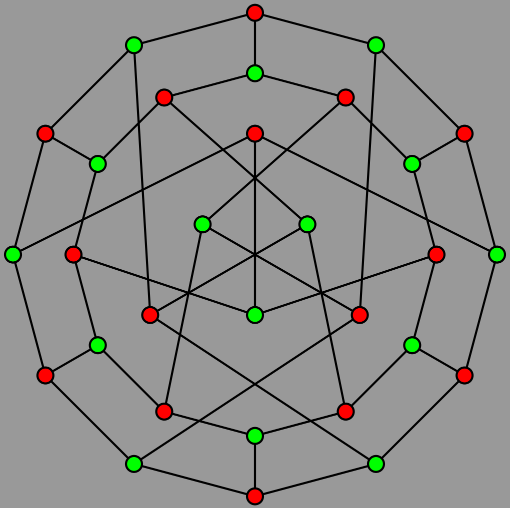

Figure: Red and green colors indicate bipartition.

$(1, 1, 6)$ Totally Regular Bipartite Graph of Order 30

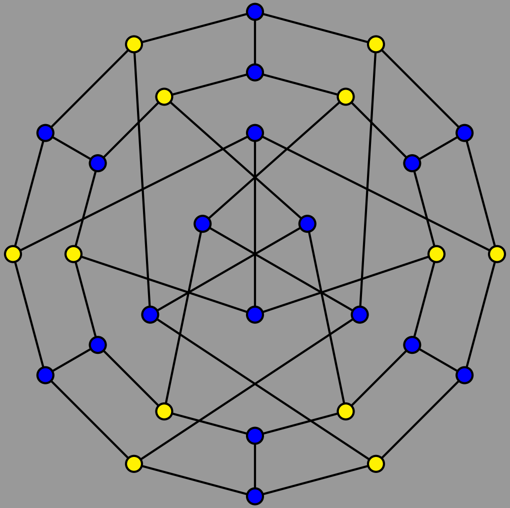

Figure: Blue and yellow colors indicate vertex orbits.

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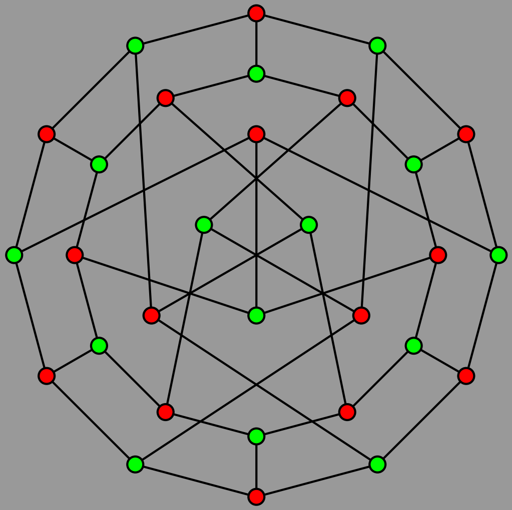

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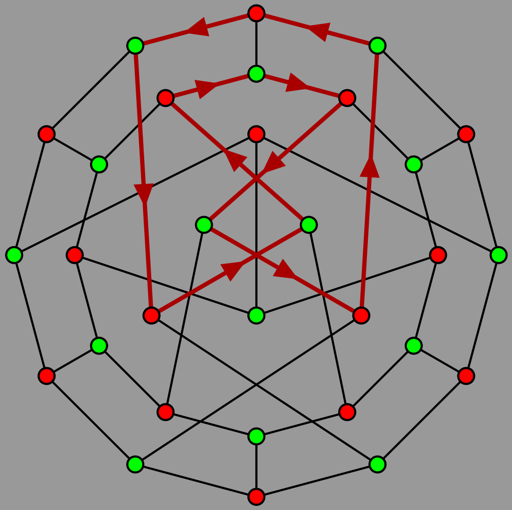

Figure: One of the oriented 10-cycles.

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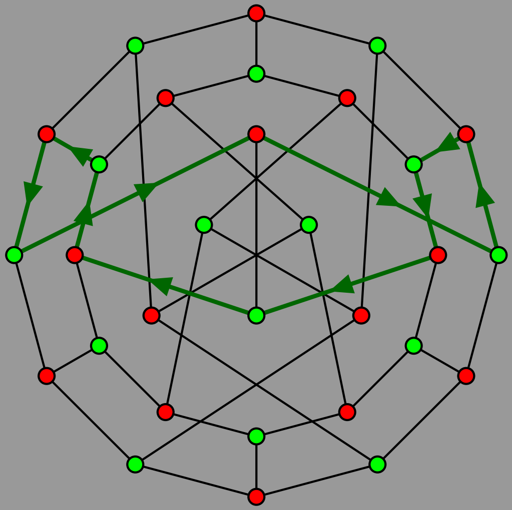

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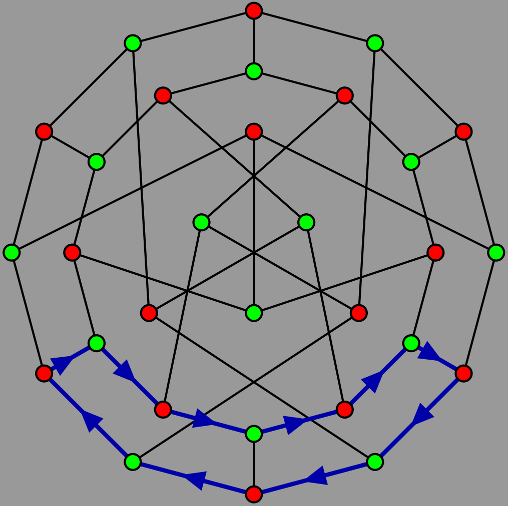

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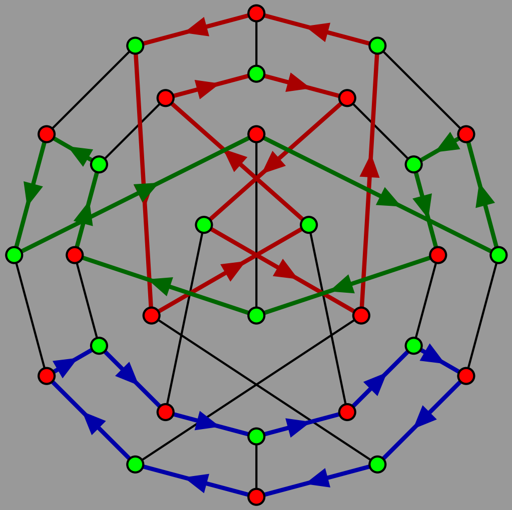

More $(1, 1, 6)$ Bipartite Graphs

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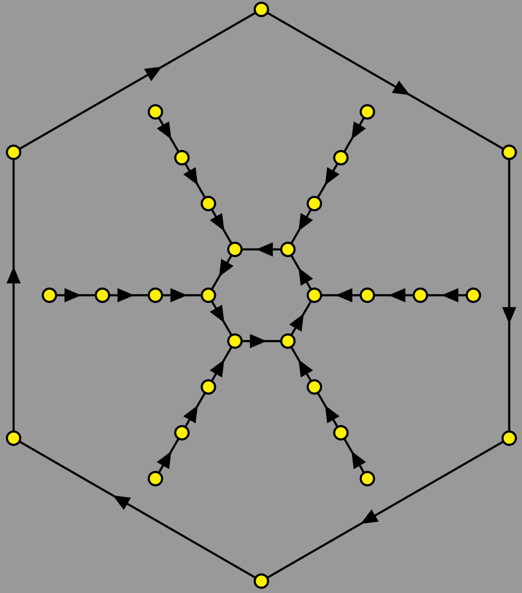

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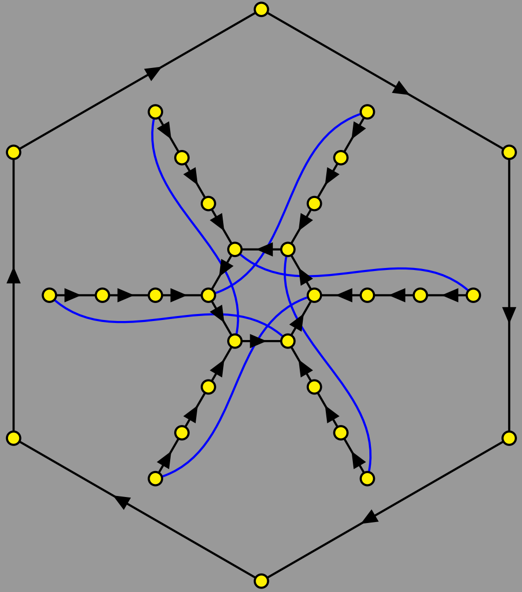

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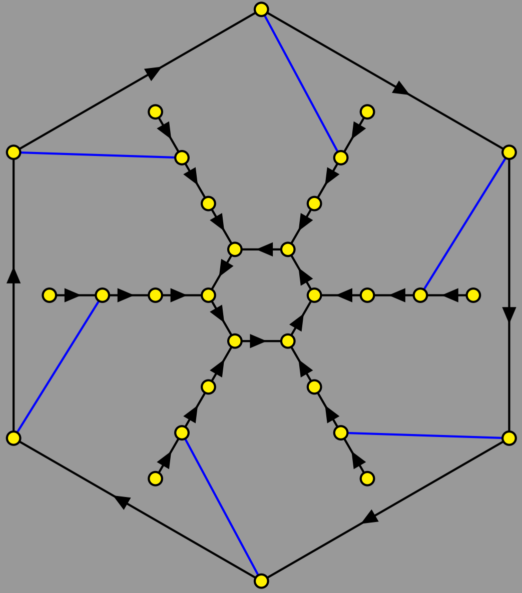

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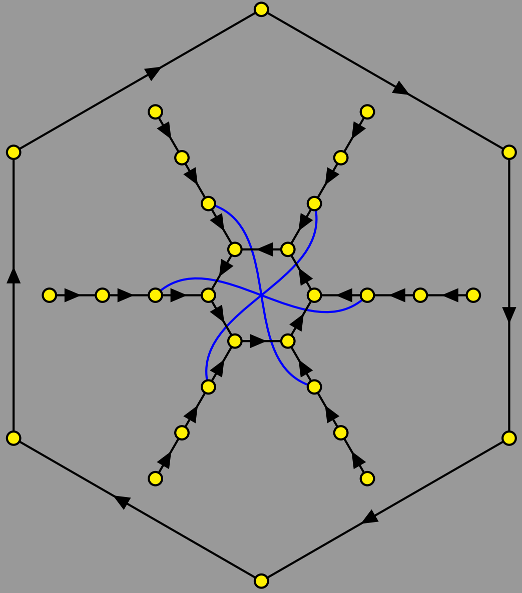

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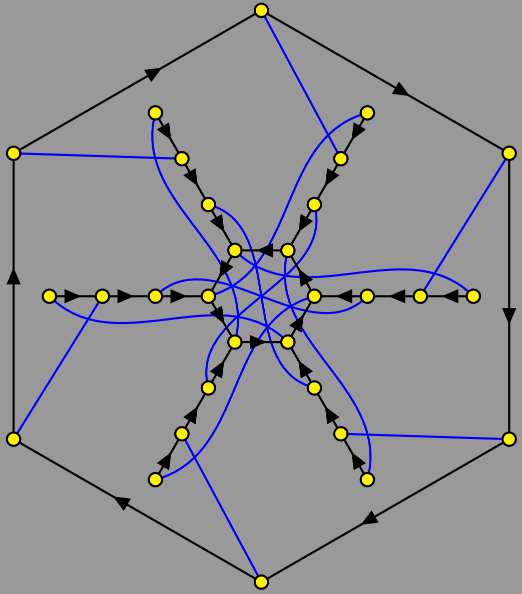

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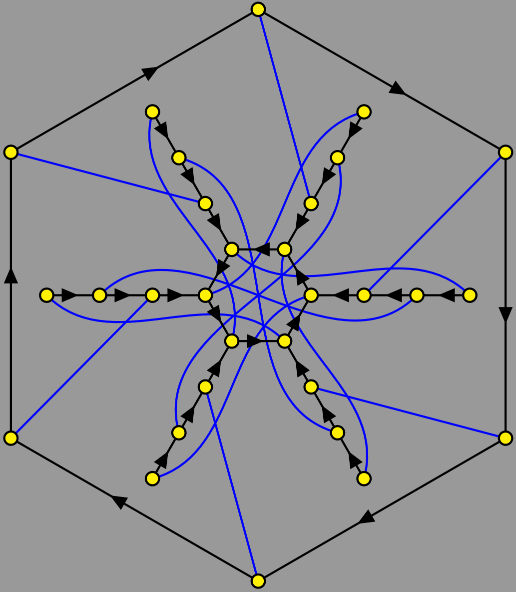

A 3D view of the second or third $(1, 1, 6)$-graph of order 30, which is not totally regular.

The Table Again

k	Moore Bound	Best Graph	Optimal?
3	8	8	Yes
4	14	12	Yes
5	24	18	Yes
6	40	30	Maybe

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- Upper bound searches are easier for bipartite graphs.

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- There are only 703 cubic bipartite graphs of order 20.
- There are only 149 cubic bipartite graphs of order 18.

Upper Bounds

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- There are only 29 579 cubic bipartite graphs of order 24.
- There are only 4 132 cubic bipartite graphs of order 22.
- There are only 703 cubic bipartite graphs of order 20.
- There are only 149 cubic bipartite graphs of order 18.
- An exact value for $k = 6$ is feasible if girth 4 and $n > 34$ can be handled separately.

Thank you.