Local spectra and symmetric powers of walk-regular graphs

M. A. Fiol
Departament de Matemàtiques, UPC
Institut de Matemàtiques de la UPC-BarcelonaTech (IMTech)

Open University Discrete Mathematics Seminar Series
April 7, 2021

Outline

Abstract

The u-local spectrum of a graph G gives similar information as the (standard) spectrum when G is 'seen' from the vertex u.
From the local spectra we can define their corresponding local characteristic functions, which can be seen as a factorization of the characteristic polynomial of G.
In this talk, we discuss some properties of the local spectra, focusing on the case of walk-regular graphs and their symmetric powers. For instance, some of the results are used to derive lower and upper bounds for the spectral radius of the token graphs, which in some cases become exact values.

Preliminaries

Graphs and spectra

Let G be a (simple and connected) graph with n vertices, adjacency matrix $\boldsymbol{A}$, and spectrum

$$\operatorname{sp} G \equiv \operatorname{sp} \boldsymbol{A} = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$. Thus, by the Perron-Frobenius theorem, G has spectral radius $\rho(G) = \theta_0$.
Let $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$ be the Laplacian matrix of G, with eigenvalues

$$\mu_1(= 0) < \mu_2 < \cdots < \mu_n.$$

Let $\phi_G(x) = \det(\boldsymbol{xI} - \boldsymbol{A})$ be the characteristic polynomial of G.
Let $\phi_{G\setminus u}(x)$ be the characteristic polynomial of the u-vertex deleted subgraph $G \setminus u$.

Symmetric powers

For a given integer k such that $1 \leq k \leq n$, the *symmetric k-power* (or *k-token graph*) $F_k(G)$ of G is the graph with:

- $V(F_k(G))$: the $\binom{n}{k}$ k-subsets of vertices of G,
- Two vertices A and B of $F_k(G)$ are adjacent whenever their symmetric difference $A \triangle B = \{a, b\}$, where $a \in A$, $b \in B$, and $\{a, b\} \in E(G)$.

In particular,

- If $k = 1$, then $F_1(G) \cong G$,
- If G is the complete graph K_n, then $F_k(K_n) \cong J(n, k)$, the (distance-regular) Johnson graph)

For more details, see Fabila-Monroy et al. (2012).

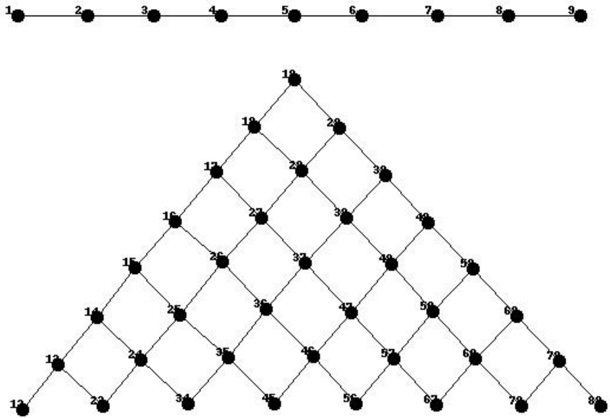


Figure: The 2-token graph $F_2(P_9)$ of the path graph P_9.

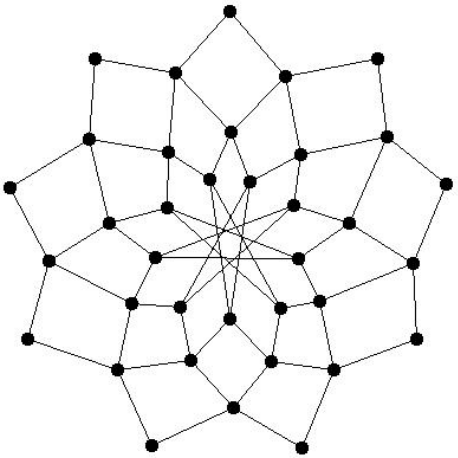


Figure: The 2-token graph $F_2(C_9)$ of the cycle graph with vertex set $V(C_9) = \mathbb{Z}_9$.

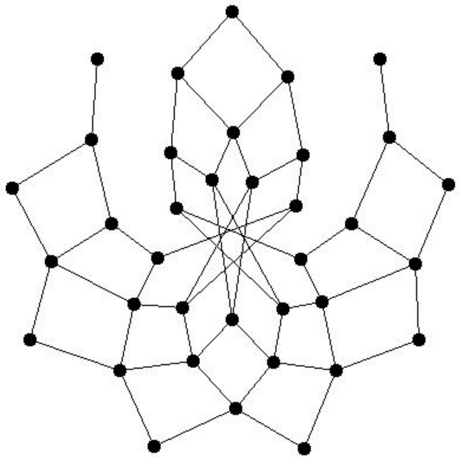


Figure: The 2-token graph $F_2(P_9)$ from $F2(C9)$.

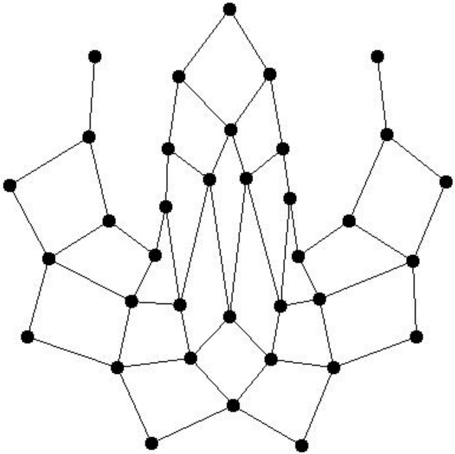
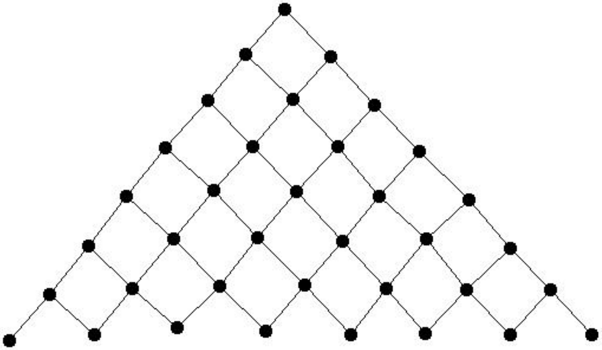


Figure: The 2-token graph $F_2(P_9)$ from $F2(C9)$.



A question

What can be said about the spectrum of $F_k(G)$?

Some results

Theorem (Audenaert, Godsil, Royle, and Rudolph (2007))

All the strongly regular graphs with the same parameters have cospectral symmetric squares (2-tokens).

Theorem (Dalfó et al. (2020))

The Laplacian spectrum of a graph on n vertices is contained in the spectrum of its k-token for every $k = 1, \ldots, n$.

Walks and characteristic polynomials

Let G be a graph with adjacency matrix $\boldsymbol{A}$, characteristic polynomial $\phi_G(x) = \det(\boldsymbol{xI} - \boldsymbol{A})$, and spectrum $\operatorname{sp} G = \{\theta_0^{m_0}, \theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$. Since $(\boldsymbol{A}^\ell)_{uv}$ is the number of ℓ-walks between vertices u and v, the matrix $\boldsymbol{W}$, whose entries are the generating functions $G_{uv}(z)$ of such numbers of walks is

$$\boldsymbol{W}_G(z) = \boldsymbol{I} + z\boldsymbol{A} + \boldsymbol{z}^2\boldsymbol{A}^2 + z^3\boldsymbol{A}^3 + \cdots = \frac{\boldsymbol{I}}{\boldsymbol{I} - z\boldsymbol{A}} = (\boldsymbol{I} - z\boldsymbol{A})^{-1}. \quad (1)$$

Then, the matrix $(\boldsymbol{I} - z\boldsymbol{A})^{-1}$ plays also an important role in determining the characteristic polynomial $\phi_{G\setminus u}$ of the u-vertex deleted subgraph $G \setminus u$. Indeed, $\phi_{G\setminus u}(x)$ is the (u,u)-entry of the adjoint matrix of $x\boldsymbol{I} - \boldsymbol{A}$:

$$\phi_{G\setminus u}(x) = \operatorname{adj}(x\boldsymbol{I} - \boldsymbol{A})_{uu} = \phi_G(x)((x\boldsymbol{I} - \boldsymbol{A})^{-1})_{uu} \quad (2)$$

A conjecture

Concerning the characteristic polynomials $\phi_{G\setminus u}$, with $u \in V(G)$, there is the long standing conjecture.

Conjecture

The characteristic polynomial ϕ_G of a graph G with at lest three vertices, is reconstructible from its polynomial deck, that is, the set of characteristic polynomials $\{\phi_{G\setminus u} : u \in V\}$.

Up to now, what is known is that ϕ_G is reconstructible from the set of pairs $(\phi_{G\setminus u}, \phi_{\overline{G}\setminus u})$ of characteristic polynomials of its vertex-deleted subgraphs and their complements (see Hagos, 2001).

The local spectra of a graph

The local multiplicities

Let G have different eigenvalues $\theta_0 > \cdots > \theta_d$, with respective multiplicities $m_0, \ldots, m_d$.
If $\boldsymbol{U}_i$ is the $n \times m_i$ matrix whose columns are the orthonormal eigenvectors of θ_i, the matrix $\boldsymbol{E}_i = \boldsymbol{U}_i^\top \boldsymbol{U}_i$, $i = 0, 1, \ldots, d$, is the *(principal) idempotent* of $\boldsymbol{A}$ and represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathrm{Ker}(\boldsymbol{A} - \theta_i \boldsymbol{I})$.
The *(u-)local multiplicities* of the eigenvalue θ_i (F. and Garriga, 1997) are defined as

$$m_u(\theta_i) = \|\boldsymbol{E}_i \boldsymbol{e}_u\|^2 = \langle \boldsymbol{E}_i \boldsymbol{e}_u, \boldsymbol{e}_u \rangle = (\boldsymbol{E}_i)_{uu} \qquad (u \in V;\ i = 0, 1, \ldots, d).$$

In particular, $m_u(\theta_0) = v_i^2 > 0$, where $\boldsymbol{v}$ is the normalized Perron eigenvector.

Why this name?

The local multiplicities are relevant when we studied the graph from a given 'base vertex'.

- $\sum_{i=0}^{d} m_u(\theta_i) = 1$,
- $\sum_{u \in V} m_u(\theta_i) = m_i$, $i = 0, 1, \ldots, d$,
- The number $a_{uu}^{(\ell)}$ of closed walks of length ℓ going through vertex u can be computed as

$$a_{uu}^{(\ell)} = \sum_{i=0}^{d} m_u(\theta_i)\theta_i^{\ell}. \qquad (3)$$

The local spectra

By picking up the eigenvalues with non-null local multiplicities, $\mu_0(=\lambda_0) > \mu_1 > \cdots > \mu_{d_u}$, say, we define the *($u$-)local spectrum* of G as

$$\operatorname{sp}_u G := \{\mu_0^{m_i(\mu_0)}, \mu_1^{m_i(\mu_1)}, \ldots, \mu_{d_u}^{m_i(\mu_{d_u})}\}$$

with *(u-)local mesh*, or set of distinct eigenvalues, $\operatorname{ev}_u G := \{\mu_0 > \mu_1 > \cdots > \mu_{d_u}\}$.

A basic property

The eccentricity of a vertex u, satisfies an upper bound similar to that satisfied by the diameter of G in terms of its distinct eigenvalues. More precisely,

$$\mathrm{ecc}(u) \leq d_u = |\,\mathrm{ev}_u\, G| - 1; \tag{4}$$

(See F., Garriga, and Yebra, 1996). In coding theory, d_u corresponds to the so-called 'dual degree' of the trivial code $\{u\}$.)

The local characteristic functions

By using the local multiplicities, $\phi_{G\setminus u}(x) = \phi_G(x)\frac{\boldsymbol{I}}{x\boldsymbol{I}-\boldsymbol{A}}$ (2) reads

$$\phi_{G\setminus u}(x) = \phi_G(x)\sum_{i=0}^{d_u}\frac{m_u(\theta_i)}{x-\mu_i}. \tag{5}$$

The aesthetic of this relation is made more apparent when given in terms of the *u-local characteristic function* $\psi_{G,u}$, defined from the local spectrum as expected; that is,

$$\psi_{G,u}(x) := \prod_{i=0}^{d_u}(x-\mu_i)^{m_u(\mu_i)}.$$

Indeed,

$$\frac{\phi_{G\setminus u}(x)}{\phi_G(x)} = \sum_{i=0}^{d} \frac{m_u(\theta_i)}{x-\theta_i} = \sum_{i=0}^{d_u} \frac{m_u(\mu_i)}{x-\mu_i} = \frac{\psi'_{G,u}(x)}{\psi_{G,u}(x)}. \qquad (6)$$

A bonus...

From (2), and adding over all the vertices, we reprove the known result

$$\sum_{u\in V} \phi_{G\setminus u}(x) = \phi_G(x) \sum_{i=0}^{d} \sum_{u\in V} \frac{m_i(\theta_i)}{x-\theta_i} = \phi_G(x) \sum_{i=0}^{d} \frac{m(\theta_i)}{x-\theta_i} = \phi'_G(x),$$

where we have used that $\sum_{u\in V} m_u(\theta_i) = m_i$.

More results

Proposition

The u-local characteristic function $\psi_{G,u}(x)$ is determined by the characteristic polynomials of G and $G \setminus u$, by the formula

$$\psi_{G,u}(x) = e^{\int \frac{\phi_{G \setminus u}}{\phi_G(x)}\, dx}. \tag{7}$$

Proposition

The characteristic polynomials of G and $G \setminus u$, for every u, are 'reconstructible' from the set of u-local characteristic functions $\{\psi_{G,u}(x) : u \in V\}$. More precisely,

$$\phi_G(x) = \prod_{u \in V} \psi_{G,u}(x) \qquad \text{and} \qquad \phi_{G \setminus u}(x) = \psi'_{G,u}(x) \prod_{v \in V \setminus u} \psi_{G,v}(x).$$

A basic lemma

Lemma

Let G be a finite graph with different eigenvalues $\theta_0 > \cdots > \theta_d$. Let $w_u^{(\ell)}$ be the number of ℓ-walks starting from (any fixed) vertex u, and let $w_{uu}^{(\ell)}$ be the number of closed ℓ-walks rooted at u. Then,

$$\rho(G) = \lim_{\ell\to\infty} \sqrt[\ell]{w_u^{(\ell)}} = \lim_{\ell\to\infty} \sup \sqrt[\ell]{w_{uu}^{(\ell)}}.$$

Proof.

We only prove that $\rho(G)$ equals the second limit (the other equality proved similarly). Using (3),

$$w_{uu}^{(\ell)} = \sum_{i=0}^{d} m_u(\theta_i)\theta_i^\ell = \theta_0^\ell \left(m_u(\theta_0) + m_u(\theta_1)\left(\frac{\theta_1}{\theta_1}\right)^\ell + \cdots + m_u(\theta_d)\left(\frac{\theta_d}{\theta_0}\right)^\ell \right)$$

$$\simeq \begin{cases} \theta_0^\ell m_u(\theta_0)[1+(-1)^\ell], & \text{if } G \text{ is bipartite,} \\ \theta_0^\ell m_u(\theta_0), & \text{otherwise,} \end{cases}$$

where we used that $m_u(\theta_0) > 0$ and $\theta_0 > |\theta_i|$ for every $i \neq 0$, except when G is bipartite, in which case $\theta_d = -\theta_0$ and $m_u(\theta_d) = m_u(\theta_0)$. Thus, the result holds by taking ℓ-th roots. □

Walk-regular graphs

Walk-regular graphs

Let $a_u^{(\ell)}$ denote the number of closed walks of length ℓ rooted at vertex u, that is, $a_u^{(\ell)} = a_{uu}^{(\ell)}$. If these numbers only depend on ℓ, for each $\ell \geq 0$, then G is called *walk-regular* (a concept introduced by Godsil and McKay (1980)).

Notice that, as $a_u^{(2)} = \delta_u$, the degree of vertex u, a walk-regular graph is necessarily regular.

Moreover, a graph G is called *spectrally regular* when all vertices have the same local spectrum: $\operatorname{sp}_u G = \operatorname{sp}_v G$ (or $\phi_{G,u} = \phi_{G,v}$) for any $u, v \in V$.

Some characterizations

Lemma

Let G be a graph with characteristic polynomial $\phi_G(x)$. Then, the following statements are equivalent.

(i) *G is walk-regular.*

(ii) *G is spectrally regular.*

(iii) *The spectra of the vertex-deleted subgraphs are all the same: $\mathrm{sp}\,(G \setminus u) = \mathrm{sp}\,(G \setminus v)$ for any $u, v \in V$.*

(iv) *For every vertex u, $\phi_{G\setminus u}(x) = \frac{1}{n}\phi'_G(x)$.*

(v) *For every vertex u, $\psi_{G,u}(x) = \phi_G(x)^{\frac{1}{n}}$.*

The spectral radius of token graphs

Two useful parameters

Let G be a graph with spectral radius $\rho(G)$ and vertex connectivity κ (that is, the minimum number of vertices whose suppression either disconnects the graph or results in a singleton).
By considering the spectral radii of its U-deleted subgraphs, with $U \subset V$, $|U| = k < \kappa$, we define the two following parameters:

$$\rho_M^k(G) = \max\{\rho(G \setminus U) : U \subset V, |U| = k\};$$
$$\rho_m^k(G) = \min\{\rho(G \setminus u) : U \subset V, |U| = k\}.$$

For particular graphs

If G is walk-regular, $\rho^1_M(G) = \rho^1_m(G) = \rho(G \setminus u)$ for every vertex u.
If G is distance regular with degree δ, its vertex-connectivity is maximum, $\kappa(G) = \delta$ (Brouwer and Koolen (2009)
Moreover, Dalfó, E. R. van Dam, and F. (2011) showed that $\mathrm{sp}(G \setminus U)$ only depends on the distances in G between vertices of U.
Thus, for every $k \leq \delta - 1$, the computation of $\rho^k_M(G)$ and $\rho^k_m(G)$ can be drastically reduced by considering only the subsets U with different 'distance-pattern' between vertices. For instance, if G has diameter D,

$$\rho^2_M(G) = \max_{1 \leq \ell \leq D} \{\rho(G \setminus \{u, v\}) : \mathrm{dist}_G(u, v) = \ell\},$$
$$\rho^2_m(G) = \min_{1 \leq \ell \leq D} \{\rho(G \setminus \{u, v\}) : \mathrm{dist}_G(u, v) = \ell\}.$$

For general graphs (using interlacing)

Lemma

Let G be a graph with n vertices, vertex-connectivity κ, and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then, for every $k = 1, \ldots, \kappa - 1$, we have

$$\lambda_{k+1} \leq \rho_M^k(G) \leq \lambda_1, \tag{8}$$

$$\lambda_n \leq \rho_m^k(G) \leq \lambda_{n-k}. \tag{9}$$

The spectral radius of token graphs

From the above results, Lemma 3, and the bounds for the spectral radius of graph perturbations obtained in Dalfó, Garriga, and F. (2011) and Nikiforov (2007) we have the following result.

Theorem

Let G be a graph with spectral radius $\rho(G)$ and vertex-connectivity $\kappa > 1$. Given an integer k, with $1 \le k < \kappa$, let $\rho_M^k(G)$ and $\rho_m^k(G)$ the maximum and minimum of the spectral radii of the U-deleted subgraphs of G where $|U| = k$. Then,

(i) *The spectral radius of the k-token graph $F_k(G)$ satisfies*

$$k\rho_m^{k-1}(G) \le \rho(F_k(G)) \le k\rho_M^{k-1}(G). \tag{10}$$

(ii) *If G is a regular graph of order n and diameter D, the spectral radius of the k-token graph $F_k(G)$ satisfies*

$$\rho(F_k(G)) \le k\left(\rho(G) - \frac{1}{nD}\right). \tag{11}$$

(iii) If $k = 2$ and G has minimum degree δ, then

$$\rho(F_2(G)) \geq 2\left(\rho(G) - \frac{\delta}{\rho(G)}\right). \tag{12}$$

(iv) If G is walk-regular and $k = 2$, we have

$$\rho(F_2(G)) = 2\rho_m^1(G) = 2\rho_M^1(G). \tag{13}$$

Some consequences

Since the different eigenvalues of the path P_n on n vertices are

$$\theta_i = 2\cos\left(\frac{i\pi}{n+1}\right) \qquad \text{for } i = 1, \ldots, n,$$

and the spectral radius of the complete bipartite graph is $\rho(K_{m,n}) = \sqrt{mn}$, we have:

Corollary

Let P_n and C_n be, respectively, the path and cycle graph on n vertices. Then,

(i) $\rho(F_2(P_n)) \leq 4\cos(\pi/n)$,

(ii) $\rho(F_2(C_n)) = 4\cos(\pi/n)$,

(iii) $\rho(F_2(K_{n,n})) = 2\sqrt{n(n-1)}$.

Some examples

n	3	4	$\cdots$	8	9	10	11
$\rho(P_{n-1})$	1	1.41421	$\cdots$	1.84776	1.87938	1.92113	1.91898
$\rho(F_2(C_n))$	2	2.82842	$\cdots$	3.69552	3.75877	3.84226	3.83796

Table: Spectral radii of the 2-tokens of the cycles from the spectral radius of the paths

Notice that:

- $F_2(C_3) = C_3 = K_3$ with spectrum $-1^{[2]}, 2,$
  whereas P_2 has spectrum $-1, 1$
- $F_2(C_4) = K_{2,4}$ with spectrum $-2\sqrt{2}, 0^{[6]}, 2\sqrt{2},$
  whereas P_3 has spectrum $-\sqrt{2}, 0, \sqrt{2}$

About infinite graphs

Corollary

Let P_∞ and C_∞ be, respectively, the infinite path and cycle graph. Then,

(i) $\rho(F_2(P_\infty)) = 4$,

(ii) $\rho(F_2(C_\infty)) = 4$.

Bibliography

1. K. Audenaert, C. Godsil, G. Royle, and T. Rudolph, Symmetric squares of graphs, *J. Combin. Theory B* **97** (2007) 74–90.
2. C. Dalfó, F. Duque, R. Fabila-Monroy, M. A. Fiol, C. Huemer, A. L. Trujillo-Negrete, F. J. Zaragoza Martínez, On the Laplacian spectra of token graphs, submitted, 2020, `https://arxiv.org/abs/2012.00808`.
3. C. Dalfó, E. R. van Dam, and M. A. Fiol, On perturbations of almost distance-regular graphs, *Linear Algebra Appl.* **435** (2011), no. 10, 2626–2638.
4. R. Fabila-Monroy, D. Flores-Peñaloza, C. Huemer, F. Hurtado, J. Urrutia, and D. R. Wood, Token graphs, *Graphs Combin.* **28** (2012), no. 3, 365–380.
5. C.D. Godsil and B.D. McKay, Feasibility conditions for the existence of walk-regular graphs, *Linear Algebra Appl.* **30** (1980) 51–61.
6. V. Nikiforov, The spectral radius of subgraphs of regular graphs, *Electron. J. Combin.* **14** (2007), #N20.

Thanks for your attention