Helly numbers of exponential lattices

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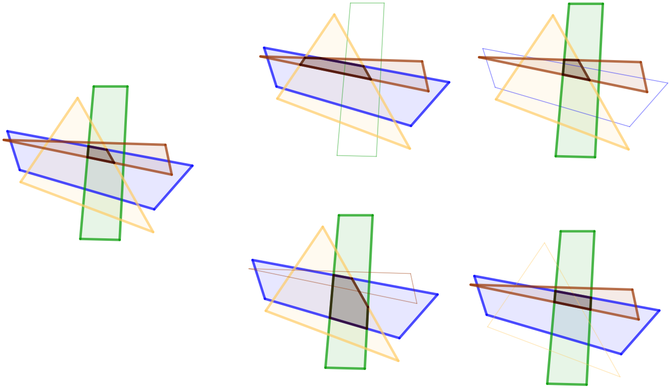
Helly's theorem

$\mathcal{F}$: a finite family of convex sets in $\mathbb{R}^d$

Helly's theorem 1913: If any $d+1$ members of $\mathcal{F}$ has non-empty intersection, then $\bigcap \mathcal{F} \neq \emptyset$



Helly numbers with respect to a set

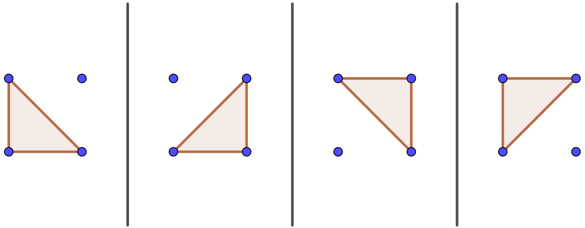
$\mathcal{F}$: a finite family of convex sets in $\mathbb{R}^d$

$H(S)$: smallest k such that: if the intersection of any k members of $\mathcal{F}$ contains a point of S, then $\cap\mathcal{F}$ also contains a point of S.

Helly's theorem 1913: $H(\mathbb{R}^d) = d + 1$

Doignon-Bell-Scarf 1973, 1977: $H(\mathbb{Z}^d) = 2^d$

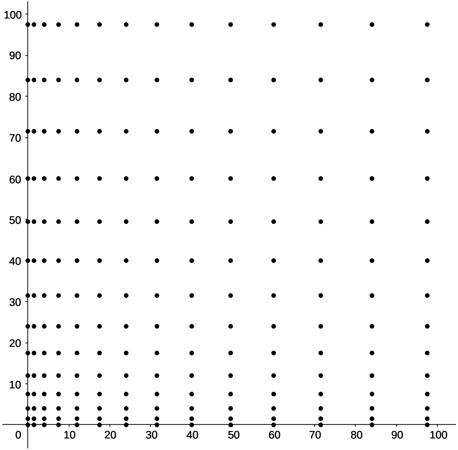
For $H(\mathbb{Z}^d) \geq 2^d$: take $Q = \{0, 1\}^d$, and $\mathcal{F} = \{\mathrm{conv}(Q \setminus \{x\}) : x \in Q\}$



Other lattices

Problem: What is $H(S)$ for various product sets $S = A^d$?

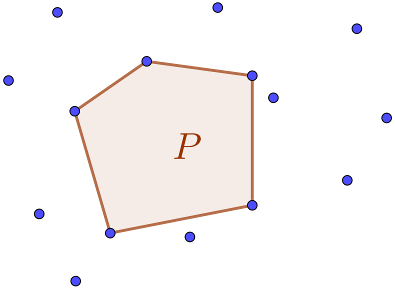
Dillon 2021: $H(A^d) = \infty$ for $A = \{p(n) : n \in \mathbb{N}\}$, where p is a polynomial of degree ≥ 2.



$$A = \left\{ \frac{n^2}{2} + n : n \in \mathbb{N} \right\}$$

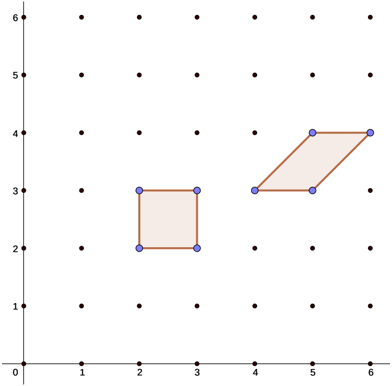
Empty polygons

Empty polygon in S: Vertices of P are from S, no other points of S inside P

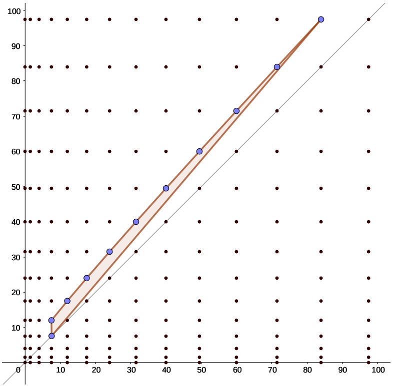


Hoffman 1979: $H(S)$=number of vertices of largest empty polygon in S

Empty polygons in lattices



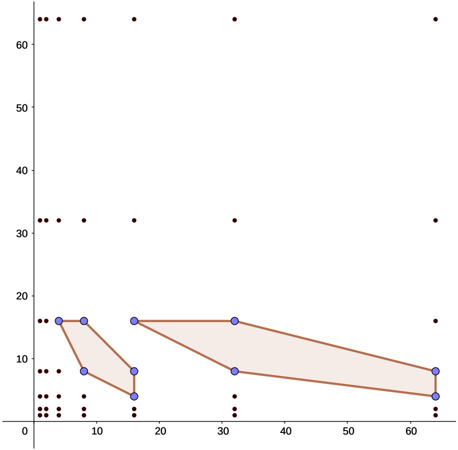
Doignon-Bell-Scarf: $H(\mathbb{Z}^2) = 4$



Dillon: $H(\{p(n)\}^2) = \infty$

Exponential lattices

Question (Dillon): What is $H(A^2)$ for $A = \{2^n : n \in \mathbb{N}\}$?

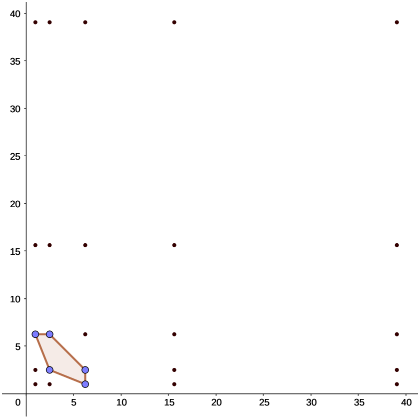


ABFJN: $H(A^2) = 5$

Larger base

$H(\alpha)=H(\{\alpha^n : n \in \mathbb{N}\}^2)$

Question: How $H(\alpha)$ changes if we change (increase) α?



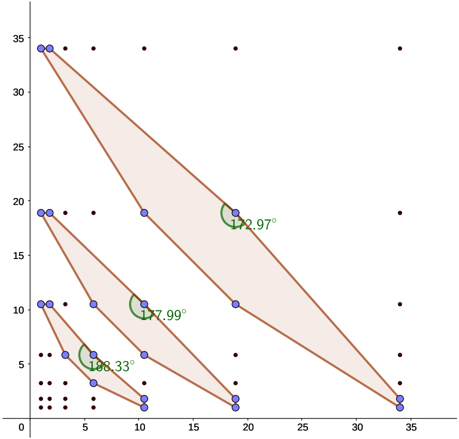
$\alpha = 2.5$

ABFJN: $H(\alpha) = 5$ for $\alpha \geq 2$

Smaller base

$H(\alpha) = H(\{\alpha^n : n \in \mathbb{N}\}^2)$

Question: How $H(\alpha)$ changes if we slightly decrease α?



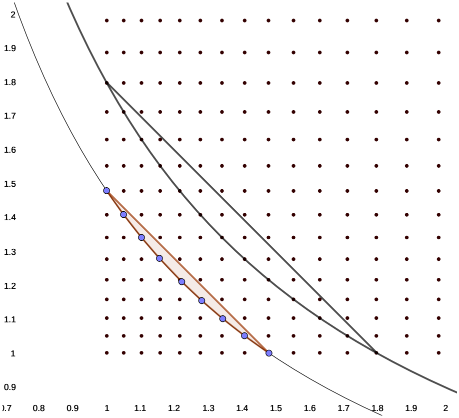
$\alpha = 1.8$

ABFJN: $H(\alpha) = 7$ for $\alpha \in [\frac{1+\sqrt{5}}{2})$

Very small base

$H(\alpha) = H(\{\alpha^n : n \in \mathbb{N}\}^2)$

Question: How $H(\alpha)$ changes if α gets close to 1?



$\alpha = 1.05$

ABFJN: $H(\alpha)$ is bounded by a function of α, but can get arbitrarily large.

Summary on $H(\alpha)$

$H(\alpha) = H(\{\alpha^n : n \in \mathbb{N}\}^2)$

ABFJN:

$H(\alpha) = 5$ for $\alpha \geq 2$

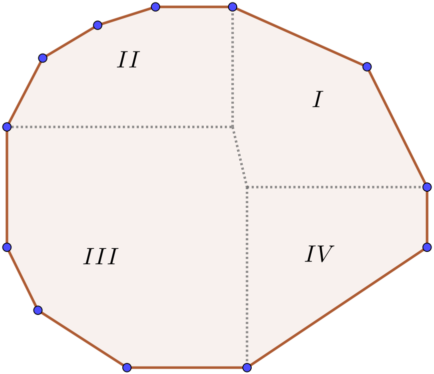
$H(\alpha) = 7$ for $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$

$\left\lfloor \sqrt{\frac{1}{\alpha-1}} \right\rfloor \leq H(\alpha) \leq 3 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1} \right) \right\rceil + 3$ for $\alpha \in (1, \frac{1+\sqrt{5}}{2})$

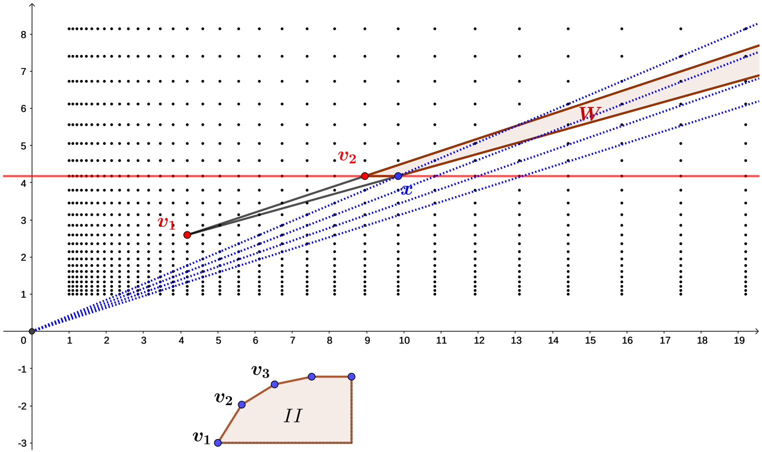
With $\alpha = 1 + \frac{1}{x}$: $\lfloor \sqrt{x} \rfloor \leq H(1 + \frac{1}{x}) \leq O(x \log_2(x))$

Upper bounds on $H(\alpha)$

Bound four different type of arcs separately:



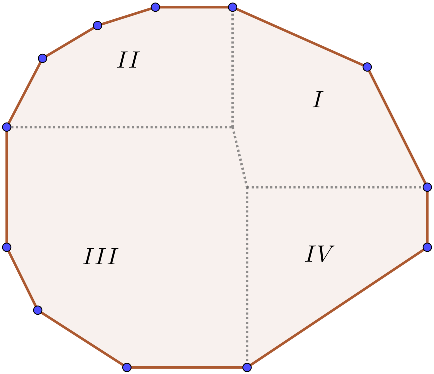
Bounding type II arcs



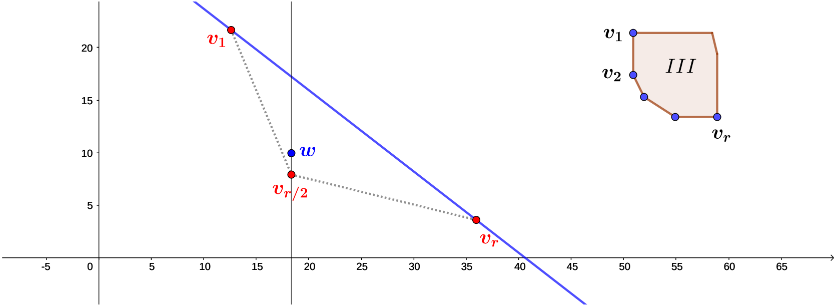
$v_3, v_4, \ldots$ must be in W

points in W can be covered by $\left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right) \right\rceil$ many lines through 0

Each line contains at most 2 points of P, so a tpye II arc has $\leq 2\left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right) \right\rceil + 2$ vertices


II
I
III
IV

Bounding type *III* arcs



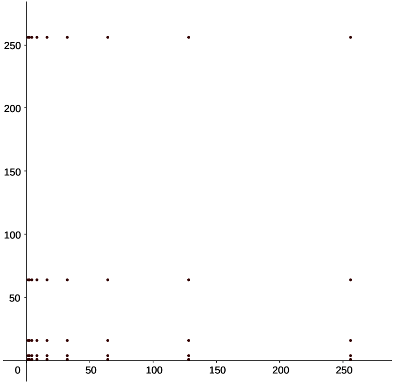
$x(w) \le \frac{x(v_r)}{\alpha^{r/2}}$, and $y(w) \le \frac{y(v_1)}{\alpha^{r/2-1}}$

$x(v_1) \le \frac{x(v_r)}{\alpha^{r}}$, and $y(v_r) \le \frac{y(v_1)}{\alpha^{r}}$

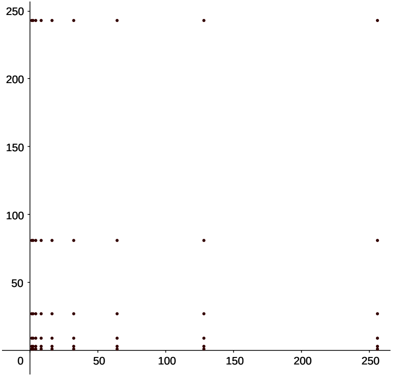
If $r \ge 2\left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right)\right\rceil + 1$, then w is below $\overline{v_1 v_r}$

Asymmetric lattices

$H(\alpha, \beta) = H(\{\alpha^n : n \in \mathbb{N}\} \times \{\beta^n : n \in \mathbb{N}\})$



$\{2^n\} \times \{4^n\}$



$\{2^n\} \times \{3^n\}$

Question: Is $H(\alpha, \beta)$ always bounded in terms of α and β?

Asymmetric lattices

ABFJN: $H(\alpha, \beta)$ is finite if and only if $\log_\alpha(\beta)$ is rational.
If $\log_\alpha(\beta) \in \mathbb{Q}$, and $\beta = \alpha^{p/q}$ then

$$\left\lfloor \frac{1}{pq} \left\lfloor \sqrt{\frac{1}{\alpha^{1/q} - 1}} \right\rfloor \right\rfloor \leq H(\alpha, \beta) \leq pq \cdot H(\alpha^p) + 1.$$

Bounds for rational $\log_\alpha(\beta)$ reduce back to the diagonal case.

Irrational $\log_\alpha(\beta)$: Construction of large empty polygons uses continued fractions and Diophantine approximation.

Continued fractions

r is irrational

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_0; a_1, a_2, a_3, \ldots]$$

Convergents of r: $\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$

Example: $\log_2(3) = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, \ldots]$ and $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}_0} = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \frac{84}{53}, \frac{485}{306}, \ldots\right)$

Simple fact: $\frac{p_n}{q_n} < r$ for n even and $\frac{p_n}{q_n} > r$ for n odd.

$\frac{p}{q}$ is a **best approximation** of r, if for any $\frac{p'}{q'} \neq \frac{p}{q}$ with $q' < q$ we have

$$\left|q'r - p'\right| > |qr - p| .$$

Continued fractions

r irrational, $r = [a_0; a_1, a_2, \dots]$, **convergents:** $\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$

Simple fact: $\frac{p_n}{q_n} < r$ for n even and $\frac{p_n}{q_n} > r$ for n odd.

$\frac{p}{q}$ is a **best approximation** of r, if for any $\frac{p'}{q'} \neq \frac{p}{q}$ with $q' < q$ we have $|q'r - p'| > |qr - p|$.

A classical result: The set of best approximations of r consists exactly of the the convergents $\frac{p_n}{q_n}$ of r.

r is **restricted** if there is an M such that $a_i \leq M$ for every i

r is **badly approximable** if there is a $c > 0$ such that for every $\frac{p}{q} \in \mathbb{Q}$ we have $\left| r - \frac{p}{q} \right| > \frac{c}{q^2}$.

Fact: restricted=badly approximable

Construction for restricted $log_\alpha(\beta)$

$\log_\alpha(\beta) = [a_0; a_1, a_2, a_3, \ldots]$, $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$

$v_n = (\alpha^{p_n}, \beta^{q_n})$, $P = \mathrm{conv}\{v_n : n \geq N, n \text{ odd}\}$ $\leftarrow$ large empty polygon

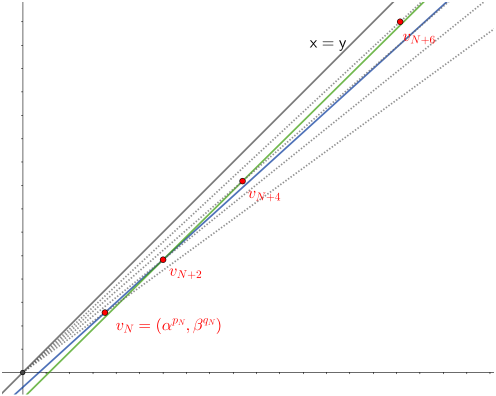
$\left(\frac{\alpha^{p_n}}{\beta^{q_n}}\right)_{n \in \mathbb{N}_0} \to 1$, for odd n: $\frac{\alpha^{p_n}}{\beta^{q_n}} > 1$

Large: $v_N, v_{N+2}, \ldots$ vertices

Slopes of edges increase

Use bad approx. and $a_i \leq M$

Empty: Use best approx.



Open problems

Problem: Find Helly numbers of diagonal lattices in higher dimension.

Conjecture (De Loera, La Haye, Oliveros, and Roldán-Pensado): $H(\mathcal{P}^2) = \infty$ for the set of prime numbers $\mathcal{P}$.

Summers: $H(\mathcal{P}^2) \geq 14$

Thank you!