On Cherlin's Conjecture

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Joint with
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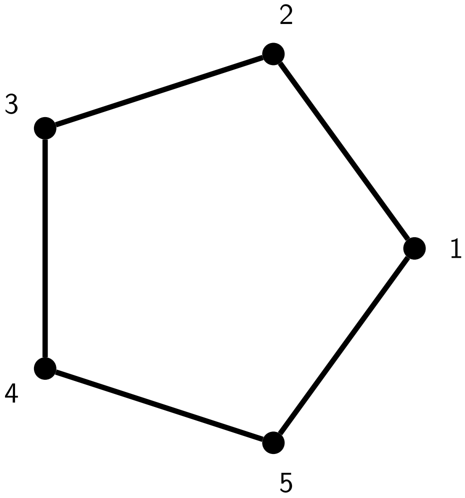
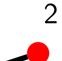
Overview

Group action 1: D_{10} on 5 points





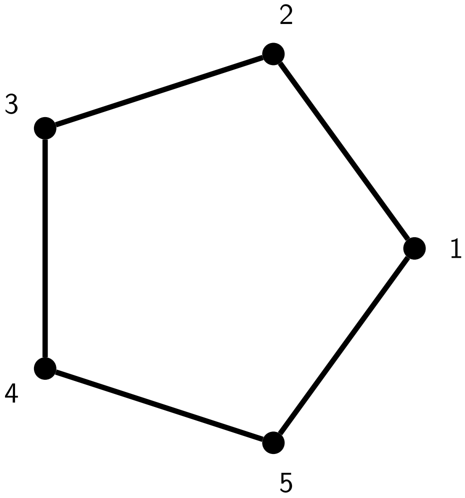
Group action 1: D_{10} on 5 points

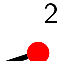
1. $Aut(\mathcal{G}) = D_{10}$.
2. $G \leq \mathrm{Sym}(\{1, 2, 3, 4, 5\})$.
3. $\mathrm{edges} \to \mathrm{edges}$

Group action 1: D_{10} on 5 points

1. The action of D_{10} on $\mathcal{G}$ is **homogeneous**.
2. Any local symmetry extends to a global symmetry.
3. Homogeneous: if $\mathcal{H}_1$ and $\mathcal{H}_2$ are induced subgraphs of $\mathcal{G}$ and $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is an isomorphism, then there exists $\sigma \in Aut(\mathcal{G})$ such that $\sigma|_{\mathcal{H}_1} = \phi$.

Group action 2: C_5 on 5 points





Group action 2: C_5 on 5 points

1. $G = C_5$ is $\mathrm{Aut}(\mathcal{S})$ where...
2. $\mathcal{S} = (\Omega, R_1, R_2)$ and R_1 is the set of 2-tuples corresponding to black edges; R_2 is the set of 2-tuples corresponding to green edges:

$$R_1 = \{(1,2),(2,3),(3,4),(4,5),(5,1)\}$$
$$R_2 = \{(1,3),(2,4),(3,5),(4,1),(5,2)\}$$

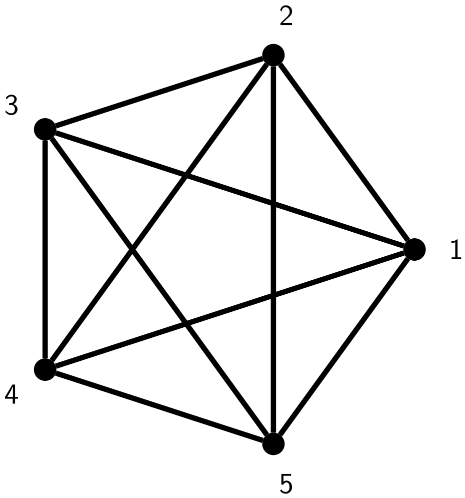
3. $\text{black-edges} \longrightarrow \text{black-edges}$
   $\text{green-edges} \longrightarrow \text{green-edges}$
4. i.e. R_1 and R_2 are both preserved by G.

Challenge

Given a permutation group G on n points, can I find a relational structure $\mathcal{S}$ such that

1. $G = \mathrm{Aut}(\mathcal{S})$;
2. $\mathcal{S}$ is homogeneous.

Group action 3: S_5 on 5 points



Group action 3: S_5 on 5 points

1. S_5 is 2-transitive so we must either have all edges of any colour or none.
2. What about other groups that are 2-transitive? For example, consider A_5 or $C_5 \rtimes C_4$.
3. These permutation groups cannot be the automorphism group of a graph with different coloured edges.

Groups actions 4 and 5 on 5 points

1. For $G = C_5 \rtimes C_4$ we can find a homogeneous **relational structure** $\mathcal{S}$ such that $\mathrm{Aut}(\mathcal{S}) = G$ if we are prepared to use 3-edges, i.e. sets of 3-tuples.
2. For $G = A_5$ we can find a homogeneous **relational structure** $\mathcal{T}$ such that $\mathrm{Aut}(\mathcal{T}) = G$ if we are prepared to use 4-edges, i.e. sets of 4-tuples.
3. In each case we end up with a relational structure $\mathcal{S} = (\Omega, R_1, \dots, R_k)$ where, for each $i = 1, \dots, k$, there is an integer $\ell_i \geq 2$ such that R_i is a set of ℓ_i-tuples with entries in Ω.
4. The R_i are **relations** and ℓ_i is their **arity**: we think of them as ℓ_i-directed edges.
5. An automorphism of such a relational structure is an element of $\mathrm{Sym}(\Omega)$ that preserves $R_1, \dots, R_k$.

Theorem

Let G be a permutation group on $\Omega = \{1, \ldots, n\}$. Then there is a homogeneous relational structure $\mathcal{S}$ on Ω such that $G = \mathrm{Aut}(\mathcal{S})$.

Proof.

Define $\mathcal{S}$ to be the relational structure $(\Omega, O_1, \ldots, O_k)$ where $O_1, \ldots, O_k$ are the orbits of G on $(n-1)$-tuples. □

Representing G efficiently

Given a permutation group G on a set Ω we want to view G as $\mathrm{Aut}(\mathcal{S})$ for some homogeneous relational structure $\mathcal{S}$... and we want to do this efficiently.

1. Let $\mathcal{S} = (\Omega, R_1, \dots, R_k)$.
2. We want to choose $\mathcal{S}$ so that its *arity*, ℓ, is minimised – this is the maximum arity of the relations $R_1, \dots, R_k$.
3. Among $\mathcal{S}$ of minimum arity, we then want to minimise k, the number of relations involved.

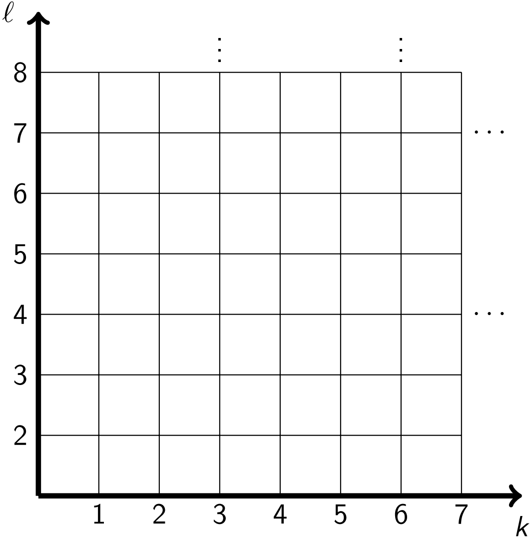
Proving theorems about finite permutation groups

Fix $k, \ell \in \mathbb{Z}^+$ with $\ell \geq 2$.

Theorem (k, ℓ)

If $\mathcal{S}$ is a homogeneous relational structure of arity at most ℓ and with at most k relations and $G = \mathrm{Aut}(\mathcal{S})$, then G is....

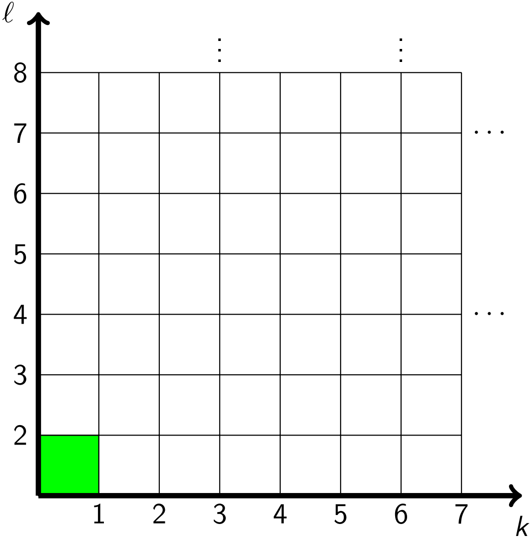
The universe of finite permutation groups



The Lachlan-Cherlin hierarchy

1. The permutation groups appearing in Theorem (k, ℓ) either belong to an infinite family corresponding to an infinite relational structure in a specific way, or else they are a sporadic example.
2. Every sporadic group in Theorem (k, ℓ) will be absorbed into an infinite family in some Theorem (k', ℓ') for some (finite) $k' \geq k$ and $\ell' \geq \ell$.
3. From the point of view of homogeneous relational structures, there are no sporadic finite permutation groups!

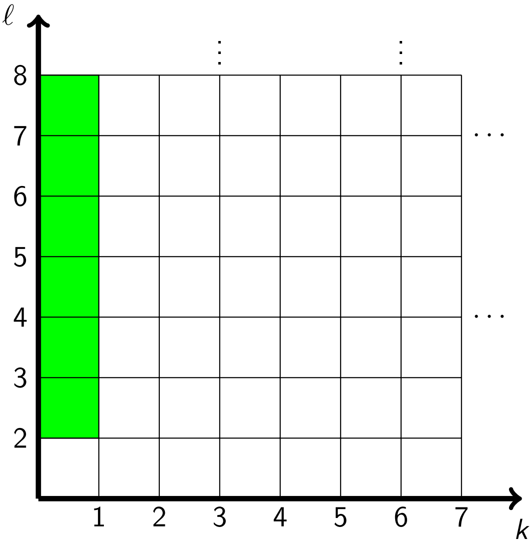
Theorem $(1,2)$



Theorem $(1, 2)$

1. This theorem describes the groups which act homogeneously on directed graphs.
2. Equivalently it is a classification of the homogeneous directed graphs.
3. This is (more or less) known thanks to work of Lachlan, building on the result for undirected graphs by Sheehan and Gardiner.
4. Gardiner's theorem describes several families and several sporadic graphs, one of which is the line graph $L(K_{3,3})$; this example turns out to belong to an infinite family occurring in Theorem $(2, 4)$ whose other members contain one relation of arity 2, and one of arity 4.

Theorem $(1, \ell)$ with $\ell > 2$



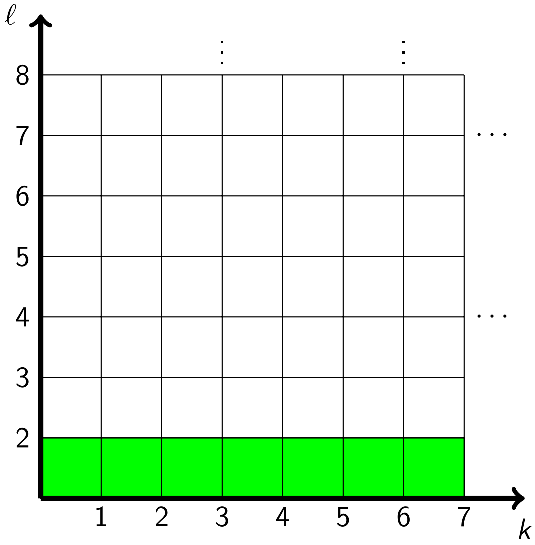
Theorems $(1, \ell)$ with $\ell > 2$

1. This theorem describes the groups which act homogeneously on directed ℓ-hypergraphs.
2. Let $\mathcal{S}$ be a directed ℓ-hypergraph for which $\mathrm{Aut}(\mathcal{S})$ acts homogeneously.
3. Observe that a set of size 2 in the vertices of $\mathcal{S}$ will induce an empty ℓ-hypergraph.
4. This means that if $\mathrm{Aut}(\mathcal{S})$ is homogeneous, then it acts 2-transitively.

Theorems $(1, \ell)$ with $\ell > 2$

1. If $\mathcal{S}$ is an **undirected** ℓ-hypergraph, then a theorem of Cameron gives the full classification.
2. **Aim**: Classify the directed ℓ-hypergraphs, thereby proving the theorems in the first column of the Lachlan-Cherlin hierarchy.

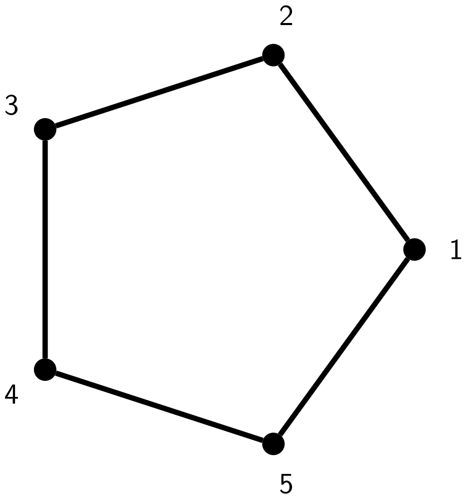
Theorems $(k, 2)$ with $k \geq 1$



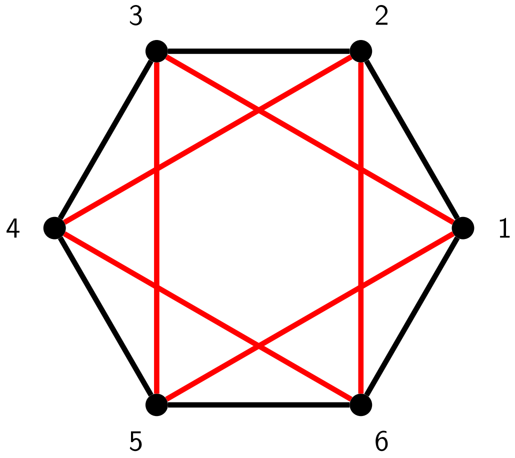
Theorems $(k, 2)$ with $k \geq 1$

1. If G is a finite permutation group, we define the **relational complexity** of G to be the lowest row in the Lachlan-Cherlin hierarchy in which G occurs.
2. If G is in the bottom row, then $\mathrm{RC}(G) = 2$ and G is called a **binary** permutation group.
3. Thus G is the automorphism group of a homogeneous relational structure for which all relations have arity 2: we can think of this as a graph with directed edges of different colours.

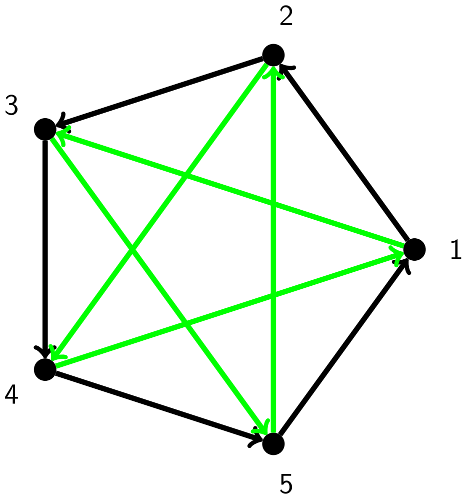
Group action 1a: D_{10} on 5 points



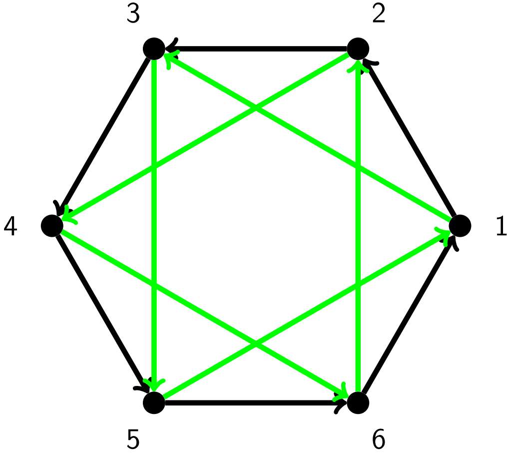
Group action 1b: D_{12} on 6 points



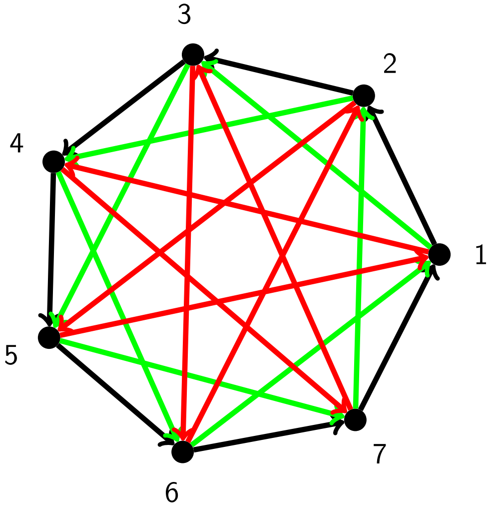
Group action 2a: C_5 on 5 points



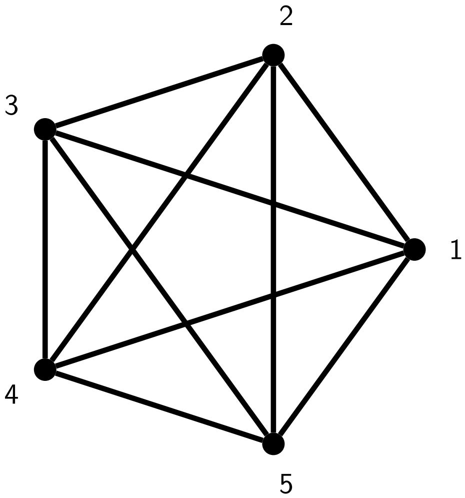
Group action 2b: C_6 on 6 points



Group action 2c: C_7 on 7 points



Group action 3a: S_5 on 5 points



Cherlin's conjecture

Cherlin has conjectured that a finite binary primitive permutation group is one of:

1. D_{2p} acting on p points with p prime;
2. C_p acting on p points with p prime;
3. S_n acting on n points;
4. $(\mathbb{F}_q, +)^2 \rtimes \mathrm{O}_2^-(q)$ acting on q^2 points with q a prime power.

Cherlin's conjecture...

...is now a theorem!

1. G. Cherlin, *On the relational complexity of a finite permutation group*, J. Alg. Combin. 2016.
2. J. Wiscons, *A reduction theorem for primitive binary permutation groups*, Bull. LMS 2016.
3. F. Dalla Volta, N. Gill, P. Spiga, *Cherlin's conjecture for sporadic simple groups*, Pac. J. Math. 2018.
4. N. Gill, F. Hunt, P. Spiga, *Cherlin's conjecture for almost simple groups of Lie rank 1*, Math. Proc. Camb. Philos. Soc. 2019.
5. N. Gill, P. Spiga, *Binary permutation groups: alternating and classical groups* Am. J. Math. 2020.
6. N. Gill, M. Liebeck, P. Spiga, *Cherlin's conjecture on finite primitive binary permutation groups*, 2021.

Beyond Cherlin's conjecture...

1. We would like to extend Cherlin's conjecture to cover imprimitive binary groups...
2. But the regular action of any finite group is binary...
3. So this is likely to be hard!

A new conjecture

Conjecture

Suppose that G is simple and G has a faithful binary action on a set of odd size. Then $G \cong \mathrm{SL}_2(2^a)$ *for some* $a \in \mathbb{Z}^+$.

Thanks for coming!