Non-Extremal Triple Arrays and Near-Triple Arrays

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Based on joint work with Lars-Daniel Öhman

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What is a triple array?

- r rows, c columns, v symbols
- no repetitions in rows or columns
- each symbol appears e times
- 2 rows, 2 columns, row and column: λ_{rr}, λ_{cc}, λ_{rc} common symbols

1	2	3	4	5	6	7	8	9
2	3	4	5	6	10	8	11	12
5	7	1	10	11	8	12	9	3
12	10	11	9	7	1	4	2	6

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1	2	3	4	5	6	7	8	9
2	3	4	5	6	10	8	11	12
5	7	**1**	10	11	8	12	9	3
12	10	11	9	7	**1**	4	2	6

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- no repetitions in rows or columns
- each symbol appears e times
- *2 rows*, 2 columns, row and column: λ_{rr}, λ_{cc}, λ_{rc} common symbols

1	2	**3**	4	**5**	6	**7**	**8**	**9**
2	3	4	5	6	10	8	11	12
5	**7**	**1**	10	11	**8**	12	**9**	**3**
12	10	11	9	7	1	4	2	6

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- r rows, c columns, v symbols
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1	2	3	4	5	6	7	8	9
2	3	4	5	6	10	8	11	12
5	**7**	1	10	11	8	12	9	3
12	10	11	9	**7**	1	4	2	6

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- 2 rows, 2 columns, *row and column*: λ_{rr}, λ_{cc}, λ_{rc} common symbols

1	2	**3**	4	5	6	7	8	9
2	**3**	**4**	5	6	10	8	**11**	12
5	7	1	10	11	8	12	9	3
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2	3	4	5	6	10	8	11	12
5	7	1	10	11	8	12	9	3
12	10	11	9	7	1	4	2	6

Why triple arrays?

- ◇ Great experimental designs
- ◇ Rich combinatorial structure; generalize latin squares, Youden rectangles

What is a triple array?

- $(r \times c, rc)$-TA

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24

- $(n \times n, n)$-TA: *latin square*

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

- $(n \times k, n)$-TA: *Youden rectangle*

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

What is a triple array?

- $(r \times c, rc)$-TA

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
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1	2	3	4	5
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- $(n \times k, n)$-TA: *Youden rectangle*

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

- $e = \lambda_{rc} = rc/v$, $\lambda_{rr} = c(e-1)/(r-1)$, $\lambda_{cc} = r(e-1)/(c-1)$

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- $(r \times c, rc)$-TA

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7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24

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1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

- $(n \times k, n)$-TA: *Youden rectangle*

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

- $e = \lambda_{rc} = rc/v$, $\lambda_{rr} = c(e-1)/(r-1)$, $\lambda_{cc} = r(e-1)/(c-1)$
- *admissible* $(r \times c, v)$: $e, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} \in \mathbb{Z}$, $\max(r,c) < v < rc$
- Ex.: $(3 \times 4, 6)$: no TA, $(5 \times 6, 10)$, $(4 \times 9, 12)$

Component designs

1	2	3	4	5	6
2	3	1	7	8	9
4	10	8	6	9	1
7	6	10	8	2	5
10	9	5	3	4	7

$(5 \times 6, 10)$-TA

Component designs

1	2	3	4	5	6
2	3	**1**	7	8	9
4	10	8	6	9	**1**
7	6	10	8	2	5
10	9	5	3	4	7

$(5 \times 6, 10)$-TA

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Column design

Component designs

1	**2**	3	4	5	6
2	3	1	7	8	9
4	10	8	6	9	1
7	6	10	8	**2**	5
10	9	5	3	4	7

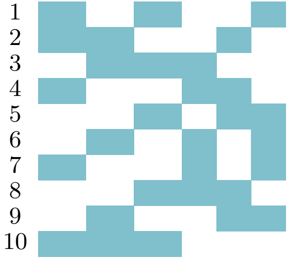
$(5 \times 6, 10)$-TA

Column design

Component designs

1	2	3	4	5	6
2	3	1	7	8	9
4	10	8	6	9	1
7	6	10	8	2	5
10	9	5	3	4	7

$(5 \times 6, 10)$-TA



Column design

Component designs

1	2	3	4	5	6
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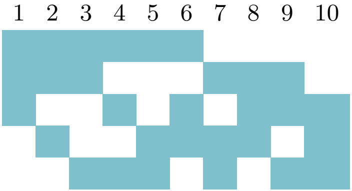
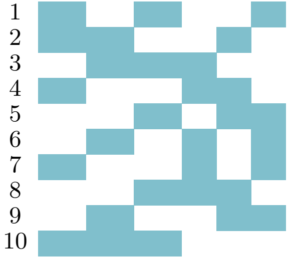


Row design



Column design

Component designs

1	2	3	4	5	6
2	3	1	7	8	9
4	10	8	6	9	1
7	6	10	8	2	5
10	9	5	3	4	7

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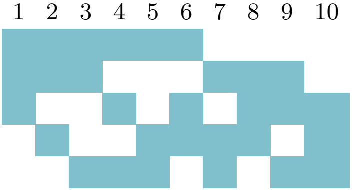

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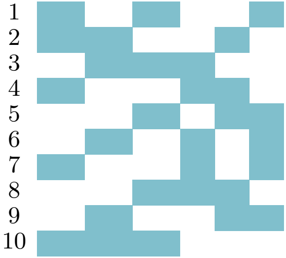

Column design

2-(v, k, λ) design: family of *blocks* (k-sets) on v points, any 2 points lie in λ blocks

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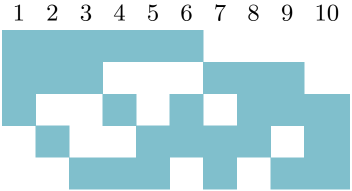

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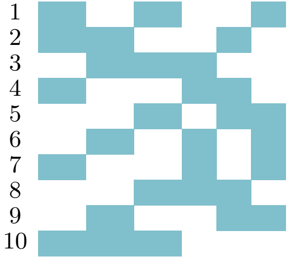

Column design

2-(v, k, λ) design: family of *blocks* (k-sets) on v points, any 2 points lie in λ blocks

$(r \times c, v)$-TA:

- ◇ row design = 2-(r, e, λ_{rr}) design
- ◇ column design = 2-(c, e, λ_{cc}) design

Extremal TA

- ◇ $v \geq r + c - 1$ **Bayley–Heidtmann, 1994; Bagchi, 1998; McSorley–Phillips–Wallis–Yucas, 2005**

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◇ symmetric 2-design $\xrightarrow[\text{problem}]{\text{assignment}}$ extremal TA **Agrawal, 1966**

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0,3,5,6,8,9

0,1,4,5,7,8

0,2,4,6,7,9

0,1,2,5,9,10

0,1,3,6,7,10

0,2,3,4,8,10

1,2,3,4,5,6						
1,2,3,7,8,9						
1,4,6,8,9,10						
2,5,6,7,8,10						
3,4,5,7,9,10						

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0,1,4,5,7,8						
0,2,4,6,7,9						
0,1,2,5,9,10						
0,1,3,6,7,10						
0,2,3,4,8,10	**1,2,4,7,10**					
1,2,3,4,5,6						
1,2,3,7,8,9						
1,4,6,8,9,10						
2,5,6,7,8,10						
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0,1,2,5,9,10

0,1,3,6,7,10

0,2,3,4,8,10	1,2,4,7,10	**2,3,6,9,10**				
1,2,3,4,5,6						
1,2,3,7,8,9						
1,4,6,8,9,10						
2,5,6,7,8,10						
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0,2,4,6,7,9
0,1,2,5,9,10
0,1,3,6,7,10

0,2,3,4,8,10	1,2,4,7,10	2,3,6,9,10	1,3,5,8,10	3,4,6,7,8	2,4,5,8,9	1,5,6,7,9
1,2,3,4,5,6						
1,2,3,7,8,9						
1,4,6,8,9,10						
2,5,6,7,8,10						
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0,2,3,4,8,10	1,2,4,7,10	2,3,6,9,10	1,3,5,8,10	3,4,6,7,8	2,4,5,8,9	1,5,6,7,9
1,2,3,4,5,6	1	2	3	4	5	6
1,2,3,7,8,9	2	3	1	7	8	9
1,4,6,8,9,10	4	10	8	6	9	1
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3,4,5,7,9,10	10	9	5	3	4	7

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0,2,3,4,8,10	1,2,4,7,10	2,3,6,9,10	1,3,5,8,10	3,4,6,7,8	2,4,5,8,9	1,5,6,7,9
1,2,3,4,5,6	1	2	3	4	5	6
1,2,3,7,8,9	2	3	1	7	8	9
1,4,6,8,9,10	4	10	8	6	9	1
2,5,6,7,8,10	7	6	10	8	2	5
3,4,5,7,9,10	10	9	5	3	4	7

- ◇ extremal TA → symmetric 2-design **BH, 1994; MPWY, 2005**

The assignment problem

		C_1	C_2	C_3	C_4	C_5	C_6
		1,2,4,7,10	2,3,6,9,10	1,3,5,8,10	3,4,6,7,8	2,4,5,8,9	1,5,6,7,9
R_1	1,2,3,4,5,6						
R_2	1,2,3,7,8,9						
R_3	1,4,6,8,9,10						
R_4	2,5,6,7,8,10						
R_5	3,4,5,7,9,10						

The assignment problem

		C_1	C_2	C_3	C_4	C_5	C_6
		1,2,4,7,10	2,3,6,9,10	1,3,5,8,10	3,4,6,7,8	2,4,5,8,9	1,5,6,7,9
R_1	1,2,3,4,5,6	$1, 2, 4$	$2, 3, 6$	$1, 3, 5$	$3, 4, 6$	$2, 4, 5$	$1, 5, 6$
R_2	1,2,3,7,8,9	$1, 2, 7$	$2, 3, 9$	$1, 3, 8$	$3, 7, 8$	$2, 8, 9$	$1, 7, 9$
R_3	1,4,6,8,9,10	$1, 4, 10$	$6, 9, 10$	$1, 8, 10$	$4, 6, 8$	$4, 8, 9$	$1, 6, 9$
R_4	2,5,6,7,8,10	$2, 7, 10$	$2, 6, 10$	$5, 8, 10$	$6, 7, 8$	$2, 5, 8$	$5, 6, 7$
R_5	3,4,5,7,9,10	$4, 7, 10$	$3, 9, 10$	$3, 5, 10$	$3, 4, 7$	$4, 5, 9$	$5, 7, 9$

- find $a_{ij} \in R_i \cap C_j$: $a_{ij} \neq a_{kj}$, $a_{ij} \neq a_{il}$

The assignment problem

		C_1	C_2	C_3	C_4	C_5	C_6
		1,2,4,7,10	2,3,6,9,10	1,3,5,8,10	3,4,6,7,8	2,4,5,8,9	1,5,6,7,9
R_1	1,2,3,4,5,6	$1,2,4$	$2,3,6$	$1,3,5$	$3,4,6$	$2,4,5$	$1,5,6$
R_2	1,2,3,7,8,9	$1,2,7$	$2,3,9$	$1,3,8$	$3,7,8$	$2,8,9$	$1,7,9$
R_3	1,4,6,8,9,10	$1,4,10$	$6,9,10$	$1,8,10$	$4,6,8$	$4,8,9$	$1,6,9$
R_4	2,5,6,7,8,10	$2,7,10$	$2,6,10$	$5,8,10$	$6,7,8$	$2,5,8$	$5,6,7$
R_5	3,4,5,7,9,10	$4,7,10$	$3,9,10$	$3,5,10$	$3,4,7$	$4,5,9$	$5,7,9$

- find $a_{ij} \in R_i \cap C_j$: $a_{ij} \neq a_{kj}$, $a_{ij} \neq a_{il}$

◇ NP-complete for arbitrary R_i, C_j **Fon-Der-Flaass, 1997**

The assignment problem

		C_1	C_2	C_3	C_4	C_5	C_6
		1,2,4,7,10	2,3,6,9,10	1,3,5,8,10	3,4,6,7,8	2,4,5,8,9	1,5,6,7,9
R_1	1,2,3,4,5,6	1, 2, 4	2, 3, 6	1, 3, 5	3, 4, 6	2, 4, 5	1, 5, 6
R_2	1,2,3,7,8,9	1, 2, 7	2, 3, 9	1, 3, 8	3, 7, 8	2, 8, 9	1, 7, 9
R_3	1,4,6,8,9,10	1, 4, 10	6, 9, 10	1, 8, 10	4, 6, 8	4, 8, 9	1, 6, 9
R_4	2,5,6,7,8,10	2, 7, 10	2, 6, 10	5, 8, 10	6, 7, 8	2, 5, 8	5, 6, 7
R_5	3,4,5,7,9,10	4, 7, 10	3, 9, 10	3, 5, 10	3, 4, 7	4, 5, 9	5, 7, 9

- find $a_{ij} \in R_i \cap C_j$: $a_{ij} \neq a_{kj}$, $a_{ij} \neq a_{il}$
- ◇ NP-complete for arbitrary R_i, C_j **Fon-Der-Flaass, 1997**
- ◇ **Conjecture:** in Agrawal's constr. solution exists if $|R_i \cap C_j| = \lambda_{rc} > 2$

Other constructions

- Extremal TA
  - from *Hadamard matrices* **Preece–Wallis–Yucas, 2005**
  - from *Youden rectangles* **Nilson–Öhman, 2014**
  - from *difference sets* **Nilson–Cameron, 2017**

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- Non-extremal TA?
  - ◇ "Small" admissible $(r \times c, v)$: $(7 \times 15, 35)$, $(11 \times 45, 99)$, $(15 \times 21, 63)$, $(16 \times 21, 56)$, $(16 \times 25, 100)$, $(13 \times 40, 130)$

◇ Is there a $(7 \times 15, 35)$-triple array? **Preece, 1970s**

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◇ Is there a $(7 \times 15, 35)$-triple array? **Preece, 1970s**

◇ There is! **MPWY, 2005; Yucas, 2002**

31	1	18	16	7	10	5	3	4	2	33	14	19	15	12
26	32	1	2	29	30	28	20	27	11	5	34	3	8	4
1	17	13	9	3	4	21	22	6	35	25	5	24	2	23
6	27	33	28	16	13	35	30	15	10	9	26	12	17	29
16	12	23	32	34	21	15	33	24	22	11	10	8	25	20
21	22	28	24	25	19	7	14	18	29	27	23	26	30	31
11	7	8	14	13	32	20	6	34	18	19	17	35	31	9

◇ The only example known so far!

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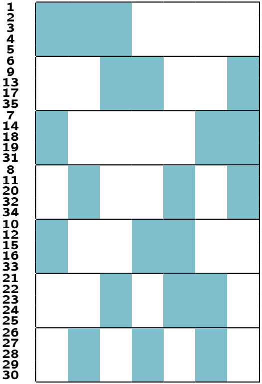
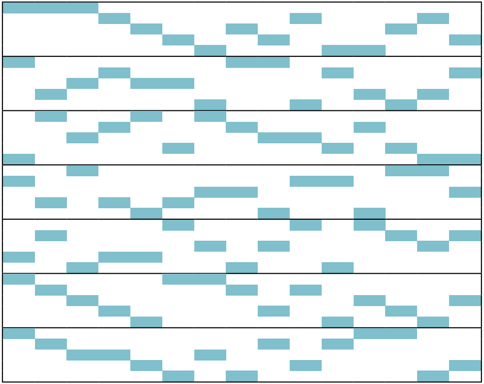
26	32	**1**	**2**	29	30	28	20	27	11	**5**	34	**3**	8	**4**
1	17	13	9	**3**	**4**	21	22	6	35	25	**5**	24	**2**	23
6	27	33	28	16	13	35	30	15	10	9	26	12	17	29
16	12	23	32	34	21	15	33	24	22	11	10	8	25	20
21	22	28	24	25	19	7	14	18	29	27	23	26	30	31
11	7	8	14	13	32	20	6	34	18	19	17	35	31	9

◇ The only example known so far!

Non-extremal $(7 \times 15, 35)$-TA



Row design: $5 \times \mathrm{PG}(2,2)$



Column design: resolution of $\mathrm{PG}(3,2)$

- *parallel class*: partition of all points into blocks
- *resolution*: partition of all blocks into parallel classes

New TA construction G.–Öhman, 2023+

- admissible $(r \times c, v)$, $a := e(e-1)/(r-1) \in \mathbb{Z}$, $k := c/e \in \mathbb{Z}$

1. row design = $k\times$ symmetric 2-(r, e, a) design
2. column design = resolution of 2-(c, e, λ_{cc}) design
3. blocks of 1 $\Leftrightarrow$ parallel classes of 2
4. assignment problem

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◇ First general construction for non-extremal TA

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- ◇ First general construction for non-extremal TA
- ◇ In constructed TA every two rows and column have a common symbols

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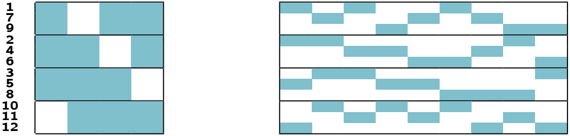
4. assignment problem

◇ First general construction for non-extremal TA

◇ In constructed TA every two rows and column have a common symbols

◇ Can give extremal TA:

1	2	3	4	5	6	7	8	9
2	3	4	5	6	10	8	11	12
5	7	1	10	11	8	12	9	3
12	10	11	9	7	1	4	2	6



New non-extremal TA

- Only one was known before!

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New $(7 \times 15, 35)$-TA

- 7 resolutions of 2-$(15, 3, 1)$ designs *(Kirkman parades)*
- Knuth's Dancing Links algorithm

◇ 85 non-isotopic $(7 \times 15, 35)$-TA **G.–Öhman, 2023+**

# of parade	1	2	3	4	5	6	7
$\lvert \mathrm{Aut} \rvert$	168	168	24	24	12	12	21
TA found	0	3	24	4	21	21	12

New non-extremal TA

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First $(21 \times 15, 63)$-TA

- 149+ resolutions of 2-$(15, 5, 6)$ designs **Mathon–Rosa, 1989**
- Exhaustive search out of the question
- Randomization + nonexhaustive techniques

$(21 \times 15, 63)$-TA

2	25	46	19	54	4	37	23	55	8	31	58	15	12	18	36	41	44	28	51	62
45	35	41	1	33	6	27	20	58	9	28	62	13	10	55	38	16	52	24	46	50
15	1	42	30	55	54	5	31	35	11	7	39	58	43	16	47	50	19	22	26	63
14	24	61	3	32	16	10	5	40	51	9	43	59	36	26	19	37	53	55	48	28
6	34	10	17	1	8	29	21	42	55	14	37	60	31	63	46	49	43	23	27	53
13	9	47	58	2	5	28	63	34	12	42	44	53	17	25	39	24	20	56	32	49
1	36	63	16	52	7	11	33	4	20	29	59	23	44	56	14	38	42	48	25	51
51	22	62	2	56	47	30	6	36	19	8	38	11	32	17	13	40	27	43	60	52
50	27	3	29	37	18	6	32	59	56	41	7	10	34	15	20	22	54	44	47	61
49	2	11	59	38	46	4	24	56	21	40	9	54	35	61	15	17	26	45	31	29
44	26	1	21	39	53	12	61	6	57	33	8	22	18	13	48	51	40	29	58	35
5	7	2	60	53	48	38	19	41	49	15	61	12	16	27	34	23	45	57	33	30
43	23	40	20	3	9	39	4	57	10	30	60	14	33	62	35	18	25	47	49	54
3	8	48	18	31	52	25	62	5	50	13	45	24	11	57	37	42	21	30	59	34
4	3	12	28	57	17	26	22	60	7	32	63	52	45	14	21	39	41	46	50	36

$(21 \times 15, 63)$-TA

2	25	46	19	54	4	37	23	55	8	31	58	15	12	18	36	41	44	28	51	62
45	35	41	**1**	33	6	27	20	58	9	28	62	13	10	55	38	16	52	24	46	50
15	**1**	42	30	55	54	5	31	35	11	7	39	58	43	16	47	50	19	22	26	63
14	24	61	**3**	32	16	10	5	40	51	9	43	59	36	26	19	37	53	55	48	28
6	34	10	17	**1**	8	29	21	42	55	14	37	60	31	63	46	49	43	23	27	53
13	9	47	58	**2**	5	28	63	34	12	42	44	53	17	25	39	24	20	56	32	49
1	36	63	16	52	7	11	33	4	20	29	59	23	44	56	14	38	42	48	25	51
51	22	62	**2**	56	47	30	6	36	19	8	38	11	32	17	13	40	27	43	60	52
50	27	**3**	29	37	18	6	32	59	56	41	7	10	34	15	20	22	54	44	47	61
49	**2**	11	59	38	46	4	24	56	21	40	9	54	35	61	15	17	26	45	31	29
44	26	**1**	21	39	53	12	61	6	57	33	8	22	18	13	48	51	40	29	58	35
5	7	**2**	60	53	48	38	19	41	49	15	61	12	16	27	34	23	45	57	33	30
43	23	40	20	**3**	9	39	4	57	10	30	60	14	33	62	35	18	25	47	49	54
3	8	48	18	31	52	25	62	5	50	13	45	24	11	57	37	42	21	30	59	34
4	**3**	12	28	57	17	26	22	60	7	32	63	52	45	14	21	39	41	46	50	36

Near-triple arrays

- r rows, c columns, v symbols
- no repetitions in rows or columns
- each symbol appears e or $e + 1$ times
- 2 rows: λ_{rr} or $\lambda_{rr} + 1$ common symbols
- 2 columns: λ_{cc} or $\lambda_{cc} + 1$ c.s.
- row and column: λ_{rc} or $\lambda_{rc} + 1$ c.s.

1	2	3	4	5	6
2	3	4	7	8	9
5	1	7	9	6	8
7	8	6	1	4	2

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1	2	**3**	4	5	6
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5	**1**	7	9	6	8
7	8	6	**1**	4	2

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1	**2**	3	**4**	5	**6**
2	3	4	**7**	**8**	**9**
5	1	**7**	**9**	6	**8**
7	8	**6**	**1**	**4**	**2**

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1	2	3	4	5	6
2	3	4	7	**8**	9
5	1	7	9	**6**	8
7	**8**	**6**	1	**4**	2

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1	2	3	**4**	5	6
2	3	4	7	8	9
5	1	7	9	6	8
7	8	6	**1**	4	2

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1	2	3	4	5	6
2	3	4	7	8	9
5	1	7	9	6	8
7	8	6	1	4	2

Why near-triple arrays?

- ◇ (Hopefully) still great experimental designs
- ◇ Exist for wider range of parameters; easier to construct
- ◇ for TA-admissible $(r \times c, v)$, TA = NTA

Existence of NTA

v	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
3 × 3	+	+	+	−	+	+	+																																									
3 × 4		+	+	−	+	+	+	+	+	+																																						
3 × 5			+	+	+	−	+	+	+	+	+	+	+																																			
3 × 6				+	+	+	+	+	+	+	+	+	+	+	+	+																																
3 × 7					+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																													
3 × 8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
3 × 9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																							
3 × 10								+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
3 × 11									+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																	
3 × 12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
3 × 13											+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+											
3 × 14												+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
3 × 15													+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
4 × 4		+	+	+	+	+	−	+	+	+	+	+	+	+																																		
4 × 5			+	+	−	+	+	−	+	+	+	+	+	+	+	+	+	+																														
4 × 6				+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
4 × 7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																						
4 × 8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																		
4 × 9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
4 × 10								+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
4 × 11									+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+						
4 × 12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
5 × 5			+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																									
5 × 6				+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
5 × 7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+															
5 × 8						+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
5 × 9							+	+	−	−	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
5 × 10								+	+	−	−	+	+	?	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6 × 6				+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
6 × 7					+	+	+	−	−	+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
6 × 8						+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
7 × 7					+	+	+	−	?	?	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	

Existence of NTA

v	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
3×3	+	+	+	−	+	+	+																																									
3×4		+	+	−	+	+	+	+	+	+																																						
3×5			+	+	+	−	+	+	+	+	+	+	+																																			
3×6				+	+	+	+	+	+	+	+	+	+	+	+	+																																
3×7					+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																													
3×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
3×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																							
3×10								+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																			
3×11									+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																	
3×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
3×13											+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+											
3×14												+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
3×15													+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
4×4		+	+	+	+	+	−	+	+	+	+	+	+	+																																		
4×5			+	+	−	+	+	−	+	+	+	+	+	+	+	+	+	+																														
4×6				+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
4×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																						
4×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																		
4×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
4×10								+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
4×11									+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+						
4×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
5×5			+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																									
5×6				+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
5×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+															
5×8						+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
5×9							+	+	−	−	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
5×10								+	+	−	−	+	+	?	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6×6				+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
6×7					+	+	+	−	−	+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
6×8						+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
7×7					+	+	+	−	?	?	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	

◇ $(3\times c, v)$-NTA exist for all $c \geq 6$

Existence of NTA

v	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
3×3	+	+	+	−	+	+	+																																									
3×4		+	+	−	+	+	+	+	+	+																																						
3×5			+	+	+	−	+	+	+	+	+	+	+																																			
3×6				+	+	+	+	+	+	+	+	+	+	+	+	+																																
3×7					+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																													
3×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
3×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																							
3×10								+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																			
3×11									+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																
3×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+													
3×13											+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+											
3×14												+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
3×15													+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
4×4		+	+	+	+	+	−	+	+	+	+	+	+	+																																		
4×5			+	+	−	+	+	−	+	+	+	+	+	+	+	+	+	+																														
4×6				+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
4×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																						
4×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																		
4×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
4×10								+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
4×11									+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+						
4×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
5×5			+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																									
5×6				+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
5×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+															
5×8						+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
5×9							+	+	−	−	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
5×10								+	+	−	−	+	+	?	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6×6				+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
6×7					+	+	+	−	−	+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
6×8						+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
7×7					+	+	+	−	?	?	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	

- $(3\times c, v)$-NTA exist for all $c \geq 6$
- $-, ? \Rightarrow +$ with just one intersection condition relaxed to x, $x+1$ or $x+2$

Enumeration of TA

	v	10	12	14	15	20
	$r \times c$	5×6	4×9	7×8	6×10	5×16
	Total #	7	1	684782	270119	26804
$\lvert \mathrm{Aut} \rvert$	1			682054	263790	26714
	2			1266	5280	
	3	2	1	1277	260	90
	4	1		98	579	
	5				1	
	6	1		48	69	
	7			2		
	8			12	88	
	10				2	
	12	2		9	17	
	16				11	
	18				1	
	20				4	
	21			8		
	24			7	9	
	36				2	
	48				4	
	60	1				
	120				1	
	168			1		
	720				1	

Open questions

- More non-extremal TA:
  - $\mathrm{PG}(2,q)$ + resolution of $\mathrm{PG}(3,q)$: $(7 \times 15, 35)$, $(13 \times 40, 130)$, $\ldots$

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◇ Are there $r, c, v \in \mathbb{N}$, $\max(r,c) < v < r + c - 1$:

$$\frac{rc}{v} \in \mathbb{N}, \ \frac{c(e-1)}{r-1} \in \mathbb{N}, \ \frac{r(e-1)}{c-1} \in \mathbb{N}?$$