An Introduction to Relational Complexity

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Overview

Group Actions

A reminder of what a group action is. Let G be a group. Let Ω be a set. A group action is a map $\phi : \Omega \times G \to \Omega$ that satisfies the following (the notation ω^g will be used for $(\omega, g)\phi$);

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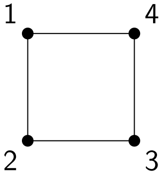
Assume all groups and all sets being acted on are finite. Also G will always be a group and Ω a set, usually acted on by G.

Action on a Square



Example

Let $\Omega = \{1, 2, 3, 4\}$ be the set of vertices of a square and let $G = D_8$, the group of order 8 consisting of reflections and rotations of the square.

These symmetries move the vertices among each other and act on Ω.

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Let $n \in \mathbb{N}$ and let $\Omega = \{1, \ldots, n\}$. The permutations of Ω form a group, S_n, known as the symmetric group.

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For example for $g = (1\ 2)(3\ 8\ 5) \in S_8$ *we have* $1^g = 2$ *and* $5^g = 3$. *If a number does not appear in a cycle then it is fixed. So* $7^g = 7$.

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An element of S_n can contain a cycle of any length up to n and the cycles can contain any numbers in any order as long as they only appear once in the cycles. This means that for any two numbers in $x, y \in \Omega$ there exists a group element that sends one to the other, such as $(x\ y)$.

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The rotations of a square form a subgroup of D_8, so it has a natural action on the set of vertices of a square.

Primitive Group Actions

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Let G be a group and $H \leq G$. A right coset of H is a set $Hg = \{hg : h \in H\}$ for some $g \in G$. The set Ω of right cosets of H is a partition of G.

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*If H is maximal, that is H is a proper subgroup and there is no other subgroup between H and G, then we say the action is **primive**. Primitive group actions have properties that tend to make them easier to work with.*

The Action on Tuples

Example

Let $n \in \mathbb{N}$ and let Ω^n be the set of n-tuples of Ω. An action on Ω^n can be defined by acting on the entries of n-tuples, i.e. $(\omega_1, \ldots, \omega_n)^g = (\omega_1^g, \ldots, \omega_n^g)$ for each $(\omega_1, \ldots, \omega_n) \in \Omega^n$ and each $g \in G$.

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More definitions are needed to make this clear.

Subtuples

Definition (Subtuple)

*Let $n \in \mathbb{N}$ and let $I \in \Omega^n$, where $I = (I_1, \ldots, I_n)$. Suppose $1 \leq m \leq n$. An m-**subtuple** of I is an m-tuple $(I_{p_1}, \ldots, I_{p_m})$ where $1 \leq p_1 < \cdots < p_m \leq n$.*

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As above, whenever $I, J \in \Omega^n$, assume the entries are written as $I = (I_1, \ldots, I_n)$ and $I = (J_1, \ldots, J_n)$.

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Example

Consider $(5, 2, 3, 8) \in \mathbb{Z}^4$. *A* 3-*subtuple of this is* $(5, 3, 8)$ *and another is* $(5, 2, 3)$. *An example of a* 2-*subtuple is* $(2, 3)$.

Subtuple Completeness

Definition (Subtuple Complete)

*Suppose G is a group acting on Ω. Let $I, J \in \Omega^n$. The pair (I, J) is m-**subtuple complete** if and only if for each corresponding pair of m-subtuples $(I_{p_1}, \ldots, I_{p_m})$ and $(J_{p_1}, \ldots, J_{p_m})$ there exists $g \in G$ such that $(I_{p_1}, \ldots, I_{p_m})^g = (J_{p_1}, \ldots, J_{p_m})$, that is $I_{p_i}^g = J_{p_i}$ for all $i \in \{1, \ldots, m\}$.*

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Notice that if $I \sim_m J$ then $I \sim_k J$ for all $1 \leq k \leq m$. This is because any corresponding pair of k-subtuples lies in a corresponding pair of m-subtuples.

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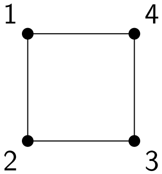

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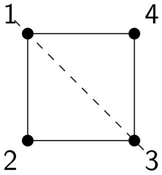

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If $I = (1, 4, 3, 2)$ and $J = (1, 2, 3, 4)$ then $I \sim_3 J$ because the reflection in a line through vertices 1 and 3 sends the entries of I to the entries of J, so sends any 3-subtuple of I to the corresponding 3-subtuple of J

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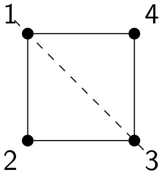

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S_6 Action

Example

Let $\Omega = \{1, 2, \ldots, 6\}$ *and let* $G = S_6$.

If $I = (1, 5, 3, 6)$ *and* $J = (2, 3, 2, 5)$ *then* $I \not\sim_2 J$. *This is because there is no permutation that sends the* 2*-subtuple* $(1, 3)$ *of* I *to the corresponding* 2*-subtuple* $(2, 2)$ *of* J.

Since these 2*-tuples lie in* 3 *and* 4*-subtuples, it follows that* $I \not\sim_3 J$ *and* $I \not\sim_4 J$.

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Following the same reasoning as above we can see that for any action, we have a pair of m-tuples (K, L) and $K \not\sim_k L$ for some $k \leq m$, then $K \not\sim_j L$ for all $j \in \{k, \ldots, m\}$.

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But would it be the case that if $K \sim_k L$ for some $k \leq m$ then $K \sim_j L$ for all $j \in \{k, \ldots, m\}$?

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This leads to the definition of relational complexity...

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Lemma

Let G be a group acting on a finite set Ω. Let $n \in \mathbb{N}$ where $n \geq 2$. Let $I, J \in \Omega^n$ where $I = (I_1, I_2, \ldots, I_n)$ and $J = (J_1, J_2, \ldots, J_n)$. Suppose $I \sim_m J$ for some $m \geq 2$. Let $i, j \in \{1, \ldots, n\}$. Then $I_i = I_j$ if and only if $J_i = J_j$.

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Lemma

Suppose G is a group acting on a set Ω. Let $g \in G$. Then g permutes the elements of Ω.

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We return to the A_4 action on $\Omega = \{1, 2, 3, 4\}$. Earlier it was shown that there exists $I, J \in \Omega^3$ such that $I \sim_2 J$ but $I \not\sim_3 J$. So the relational complexity is not 2.

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Let's test to see if it is 3. Let $m \geq 3$. Suppose $K, L \in \Omega^m$ and $K \sim_3 L$. It needs to be shown that $K \sim_m L$. Write these tuples as $(K_1, K_2, \ldots, K_m)$ and $L = (L_1, L_2, \ldots, L_m)$.

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If there are 3 or fewer distinct entries then there exists a 3-subtuple K' containing all these entries and a corresponding 3-subtuple L'. Since $K \sim_3 L$, there exists $g \in A_4$ such that $(K')^g = L'$. Using the fact repeated entries of K and L correspond, it follows that $K^g = L$.

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If there are four distinct entries in K then by the earlier lemma there are four distinct entries of L. Assume without loss of generality that these are K_1, K_2, K_3 and K_4 in K (also L_1, L_2, L_3 and L_4 in L).

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So $K \sim_3 L \Rightarrow K \sim_m L$. It follows that the action has relational complexity 3.

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Similar to how the relational complexity of the A_4 action was found, there are at most n distinct entries of I and these can be placed in a n-subtuple I'. The corresponding n-subtuple J' must contain the unique entries of J by the earlier lemma on repeating entries.

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So the relational complexity of the action exists and is at most $|\Omega|$ (or at most $|\Omega| - 1$ if $|\Omega| \geq 3$).

Relational Structures

Earlier it was mentioned that there is an alternative definition of relational complexity, which comes from model theory.

Definition (Relational complexity)

*Let G be a finite group acting non-trivially on a finite set Ω. Let K be the kernel of the action. The **relational complexity** of the action is the minimum $k \in \mathbb{N}$ such that $k \geq 2$ and there exists a relational structure $\Gamma = (\Omega, R_1, \ldots, R_r)$ of type k where*

- *$R_i \neq \emptyset$ for some $i \in \{1, \ldots, r\}$,*
- *Γ is homogeneous,*
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This is quite difficult to work with though, which is why the other subtuple definition is used.

Origins in Model Theory

The model theory definition is where interest in relational complexity comes from. A paper by Cherlin, Martin and Saracino (1996) is where it began to be looked at from a group theory point of view, although at that time it had only been calculated for a small number of actions. The topic was revisited in Cherlin (2000).

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In both of these papers it was know as "arity". The term relational complexity start be used later, again by Cherlin (2016).

Ideally we want to calculate the relational complexity of all group actions, but since this is such a large undertaking it has largely been restricted to primitive group actions so far.

Current Research

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$$GL_2(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q,\ ad - bc \neq 0 \right\}$$

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$$SL_2(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q, \ ad - bc = 1 \right\}.$$

The maximal subgroups of $SL_2(q)$ fall into six different classes and for $GL_2(q)$ there are five classes of maximal subgroups. The subgroups in $GL_2(q)$ are related to those in $SL_2(q)$, which means they can generally be dealt with together.

Current Research

In a paper by Gill, Hunt, Spiga (2019) it is shown that there is a lower bound of 3 for the relational complexity of the actions on these maximal subgroups whenever $q > 9$. All other cases, where $q \leq 9$ can be found using GAP and this has been done by Joshua Wiscons, who has provided a list for all primitive actions on sets of size at most 100 (for any group).

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I have been working through each of these classes of actions to find exact answers and shown that for three of the classes in both $SL_2(q)$ and $GL_2(q)$, the relational complexity is 3 for all but a small number of special cases.

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The remaining actions are still to be completed, but from the work done by Wiscons we can see that the relational complexity is not always 3 and is likely to be larger for some classes.

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Definition (Independent Set)

Suppose G is a group acting on a set Ω. A non-empty set $\Delta \subseteq \Omega$ is an ***independent set*** *if for any proper non-empty subset $\Gamma \subset \Delta$ we have $G_{(\Delta)} \neq G_{(\Gamma)}$. Any set of size 1 is always defined to be independent.*

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Definition (Height)

*The **height** of an action is the size of the largest independent set, denoted as $Ht(G, \Omega)$.*

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It can be shown that $RC(G,\Omega) \leq Ht(G,\Omega)+1$. So it is of interest to find the height of group actions, which I am also doing for the actions of $SL_2(q)$ and $GL_2(q)$.

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If you are familiar with the notion of a base of a group then note that the height of an action is at least as large at the size of the largest minimal base of the action.