Well quasi-order under the consecutive order: equivalence relations and permutations

Victoria Ironmonger

based on work with Nik Ruškuc

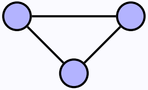
February 27, 2025

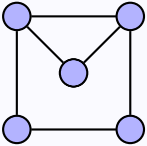
Structures and substructures

Words aab $ababa$

Permutations 132 24135

Graphs





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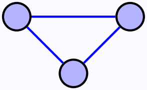
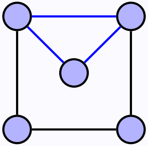
Graphs  

Two structures of the same kind will be related when one is a substructure of the other, forming a poset.

Posets

Definition

A poset is a pair $(X, \leq)$ where X is a set and $\leq$ is a reflexive, antisymmetric, transitive binary relation on X. This relation will be called an order.

Examples

1. The set A^+ with the subword order;
2. Permutations with the subpermutation order;
3. The set of finite graphs with the subgraph order;
4. etc...

Well quasi-order

Definition

An antichain is a set $\{a_1, a_2, \dots\}$ such that $a_i \not\leq a_j$ if $i \neq j$.

Eg. The permutations $1234, 132, 54321$ form an antichain.

Definition

A poset is well quasi-ordered (wqo) if it contains no infinite antichains (or infinite descending sequences).

Eg. The set of increasing permutations is wqo as it forms a chain

$$1 \leq 12 \leq 123 \leq \dots$$

so there are no antichains at all.

Avoidance sets

We consider subsets of posets, particularly downward closed subsets, in the form of avoidance sets. These are given by their forbidden substructures.

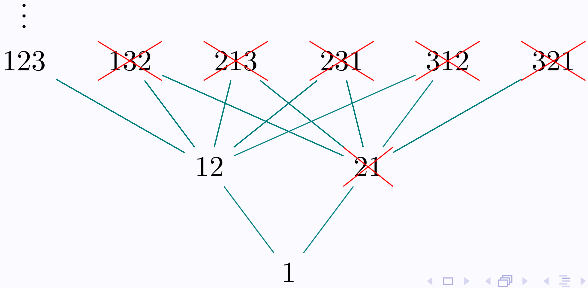
Definition

Given $(X, \leq)$ and $B \subseteq X$, the avoidance set of B is

$$\mathrm{Av}(B) = \{x \in X | b \not\leq x \quad \forall b \in B\}.$$

Example

Av(21) is the set of increasing permutations:



The wqo problem

Avoidance sets give rise to natural decidability questions: given B finite, we ask about decidability of properties of $\mathrm{Av}(B)$.

The wqo problem: For a poset $(C, \leq)$, is it decidable, given $B \subseteq C$ finite, whether $\mathrm{Av}(B)$ is wqo?

Note: if $(C, \leq)$ is wqo, its avoidance sets are also wqo so the wqo problem is trivially decidable.

Why the wqo problem?

Well quasi-order is often taken to be an indicator of the ‘wildness’ of a poset – those which are wqo are comparatively ‘tame’.

For instance, wqo posets are precisely those with countably many downward closed subsets (ideals) [Huczynska & Ruškuc, 2015].

The wqo problem asks not only whether an individual avoidance set is ‘tame’ or ‘wild’, but whether there is a clear demarkation between the ‘tame’ and ‘wild’ avoidance sets [Cherlin, 2011].

Known results

The wqo problem is decidable for:

- Graphs under the subgraph order (Ding, 1992);
- Graphs under the graph minor order (trivially decidable) (Robertson & Seymour, 2004);
- Permutations under the consecutive order (McDevitt & Ruškuc, 2021);
- Words under the non-consecutive subword order (trivially decidable) (Higman, 1952);
- Words under the consecutive subword order (McDevitt & Ruškuc, 2021).

Open questions

The wqo problem is open for:

- Permutations under the (non-consecutive) subpermutation order;
- Graphs under the induced subgraph order;
- Tournaments under the subgraph order.

Open Question

Is there an example of a natural poset of combinatorial structures for which the wqo problem is undecidable?

Consecutive orders

We will consider a variation on substructure orders, called *consecutive* orders, where we require the embedding of structures to be consecutive in some sense.

These have been studied in some depth for permutations, particularly by Elizalde (eg. Elizalde, 2016), and also for words (eg. McDevitt & Ruškuc, 2021).

This ordering has been extended to combinatorial structures more generally, and ongoing research investigates the wqo problem for these posets.

This talk will focus on two interesting cases: permutations and equivalence relations under consecutive orders.

Consecutive orders - intuition

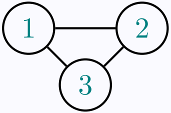
For words, the consecutive order is very natural:

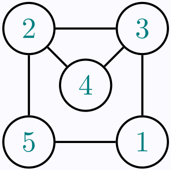
$$aab \leq baaba \qquad aab \not\leq ababa$$

Similarly, for permutations:

$$132 \leq 31254 \qquad 132 \not\leq 13524$$

For other structures, we will need an additional linear order:



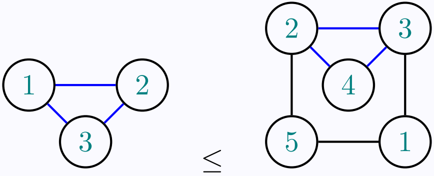


Consecutive orders - definition

Consecutive orders respect the linear order:

$A \leq B$ when there is an embedding $f : A \to B$ s.t. for some k, $f(1) = k$, $f(2) = k+1$, $f(3) = k+2$, etc.

132	$\leq$	31254
1 2 3		1 2 3 4 5
132	$\not\leq$	42513
1 2 3		1 2 3 4 5
aab	$\leq$	$baabba$
1 2 3		1 2 3 4 5 6



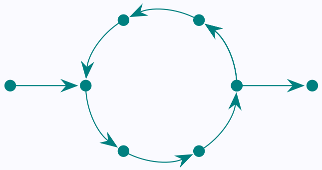
Ideas from graph theory

Definition

If η, π are paths in a finite digraph, then $\eta \leq \pi$ under the subpath order if and only if η is a subpath of π.

Definition

A cycle in a digraph is an in-out cycle if at least one vertex has in degree > 1 and at least one vertex has out degree > 1.



Theorem (McDevitt & Ruškuc, 2021)

The set of paths of a finite digraph G under the subpath order is wqo if and only if G contains no in-out cycles.

Factor graphs

We can often encode structures in our posets as paths in certain digraphs.

Consider $C = \mathrm{Av}(B)$ and take b to be the maximum size of an element in B.

The factor graph of C is the digraph Γ_C with:

- Vertices: structures of size b in $\mathrm{Av}(B)$;
- Edges: $\sigma \to \tau$ iff the last $b-1$ points of σ are isomorphic to the first $b-1$ points of τ, or formally, $\sigma\restriction_{[2,b]} \cong \tau\restriction_{[1,b-1]}$.

We will have a running example of permutations.

Factor graphs for permutations

Consider $C = \mathrm{Av}(B)$ and take b to be the maximum size of a permutation in B.

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Here is the factor graph of $\mathrm{Av}(213, 231, 312)$:

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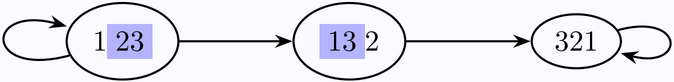

Structures trace paths

Structures in C trace paths in Γ_C: eg. 12354 traces the path 123 $\rightarrow$ 123 $\rightarrow$ 132.

The path traced by a structure σ will be denoted $\Pi(\sigma)$.

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The path traced by a structure σ will be denoted $\Pi(\sigma)$.

Paths can be traced by more than one structure

A path π in Γ_C is associated with the set
$\Sigma(\pi) = \{\sigma \in C \mid \Pi(\sigma) = \pi\}$ of structures that trace it.

Eg. 132 $\rightarrow$ 321 is traced by both 1432 and 2431.

Paths can be traced by more than one structure

A path π in Γ_C is associated with the set $\Sigma(\pi) = \{\sigma \in C \mid \Pi(\sigma) = \pi\}$ of structures that trace it.

Eg. $132 \to 321$ is traced by both $1\,432$ and $2\,431$.

Definition

A path π is ambiguous if more than one structure traces it, i.e. $|\Sigma(\pi)| > 1$.

It turns out that the wqo problem is decidable if all or no paths in a factor graph are ambiguous.

Antichains from ambiguous cycles

Lemma

If Γ_C contains an ambiguous cycle, C is not wqo.

Lemma

If all paths in Γ_C are ambiguous, C is wqo if and only if Γ_C contains no cycles, i.e. C is finite.

Theorem

The wqo problem is decidable for the following structures under consecutive orders:

1. *Graphs;*
2. *Digraphs;*
3. *Tournaments;*
4. *n-ary relations;*
5. *Collections of n binary relations.*

Antichains from in-out cycles

Lemma: If $\sigma \leq \rho$, then $\Pi(\sigma) \leq \Pi(\rho)$.

By the contrapositive: if $\Pi(\sigma) \not\leq \Pi(\rho)$ then $\sigma \not\leq \rho$.

Since a digraph contains an in-out cycle iff its paths are not wqo we have:

Lemma

If Γ_C contains an in-out cycle, C is not wqo.

Lemma

If Γ_C contains no ambiguous paths, C is wqo if and only if Γ_C contains no in-out cycles, so the wqo problem is decidable.

Segue to permutations and equivalence relations

We have seen that the wqo problem is decidable for structures when all or no paths are ambiguous in their factor graphs.

What if some paths are ambiguous, and others are not?

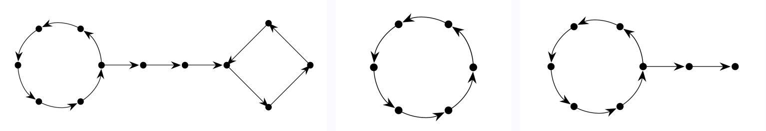
Permutations and equivalence relations are both examples of structures where this occurs.

We have already seen that in-out cycles and ambiguous cycles yield infinite antichains. Can infinite antichains arise in other ways? Yes!

Permutations - bicycles

Definition

A bicycle is a digraph consisting of two vertex disjoint, simple cycles connected by a simple path, where only the start and end vertices coincide with either cycle. The cycles may be empty, and if one cycle is absent then the connecting path may also be absent.



Theorem

If a digraph contains no in-out cycles, each of its paths is wholly contained in one of a finite number of bicycles. (McDevitt & Ruškuc, 2021)

Permutations - splittable pairs

Consider the paths that start at 132, go to 321, and then wind round the loop $n \geq 1$ times.

We can pick associated permutations for these paths with the first entry to be between the last two entries. This gives the infinite antichain $2431, 25431, 265431, \ldots$

This is an example of an infinite antichain arising from a *splittable pair*.

Theorem: An avoidance set C of permutations is wqo if and only if Γ_C contains no in-out cycles or ambiguous cycles and none of its bicycles contain a splittable pair. (McDevitt & Ruškuc, 2021)

Equivalence relations

Definition

An equivalence relation on $X = \{1, \ldots, n\}$ is a binary relation on X which is reflexive, symmetric and transitive. It partitions X into equivalence classes.

Examples

$$|1\,3|2|4| \qquad\qquad |1\,2\,4|3\,6|5|$$

Definition

Two equivalence relations are isomorphic if, when we relabel their smallest points 1, second smallest 2, etc, they are identical.

Eg. $|1\,6|3| \cong |1\,3|2| \cong |3\;9|5|$.

We consider isomorphic equivalence relations to be equal.

Consecutive order for equivalence relations

Definition

$\sigma \leq \rho$ under the consecutive order iff there is a 1-1 map $f : \sigma \to \rho$ such that:

1. If $f(1) = k$ then $f(2) = k + 1$, $f(3) = k + 2, \ldots,$ and
2. x, y are in the same class of $\sigma \iff f(x), f(y)$ are in the same class of ρ.

Example

$|\,1\,2\,|\,3\,| \leq |\,1\,2\,3\,|\,4\,5\,|$ as $f : x \mapsto x + 1$ defines a 1-1 map between them satisfying conditions (1) and (2):

$|\,1\,2\,|\,3\,| \leq |\,1\,2\,3\,|\,4\,5\,|$.

But $|\,1\,2\,|\,3\,| \not\leq |\,1\,|\,2\,4\,|\,3\,5\,|$.

Another example

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Example

$|1|2|3\,4| \leq |1\,2\,5\,6|3|4\,7|$ as the map $f : x \mapsto x+2$ gives a 1-1 map between them satisfying both conditions.

For condition (2), see that f preserves the equivalence classes:
$|\,1\,|\,2\,|\,3\;\;4\,| \leq |\,1\,2\;5\;\;6\,|\,3\,|\,4\;7\,|$.

Factor graphs for equivalence relations

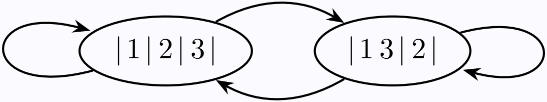
Consider $C = \mathrm{Av}(B)$ and take b to be the maximum length of an equivalence relation in B.

The factor graph of C is the digraph Γ_C with:

- Vertices: equivalence relations of length b in $\mathrm{Av}(B)$;
- Edges: $\sigma \to \tau$ iff the last $b-1$ points of σ are isomorphic to the first $b-1$ points of τ, or formally, $\sigma\restriction_{[2,b]} \cong \tau\restriction_{[1,b-1]}$.

Example

$\mathrm{Av}(|1\,2\,|\,3|, |1\,2\,3|, |1\,|\,2\,3|)$ has factor graph



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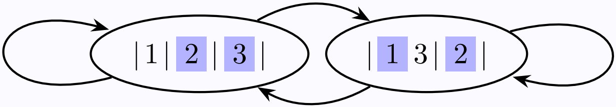
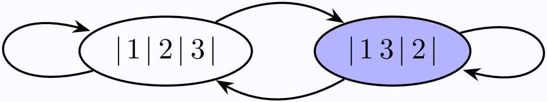


Equivalence relations trace paths



Equivalence relations in C trace paths in Γ_C:
eg. $\sigma = |\,1\;\;3\,|\,2\,|\,4|$ traces $\Pi(\sigma) = |\,1\,3\,|\,2\,| \to |\,1\,|\,2\,|\,3\,|$.

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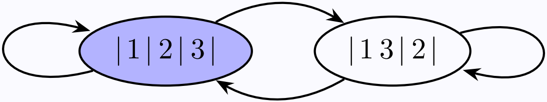

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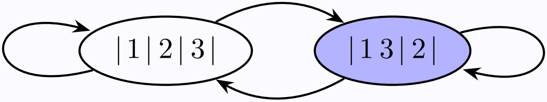

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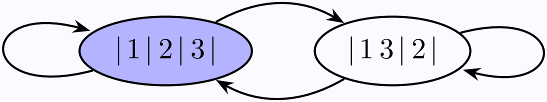
Eg. $|\,1\,3\,|\,2\,| \to |\,1\,|\,2\,|\,3\,| \to |\,1\,|\,2\,|\,3\,|$ is associated with $|\,1\;\;3\,|\,2\,|\,4\,|\,5\,|$ and $|\,1\;\;3\,|\,2\;5\,|\,4\,|$.

Equivalence relations trace paths



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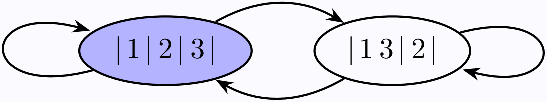
Equivalence relations trace paths



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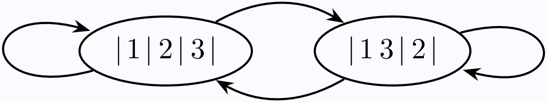
Eg. $|1\,3|2| \to |1|2|3| \to |1|2|3|$ is associated with $|1\,3|2|4|5|$ and $|1\,3|2\,5|4|$.

WQO for equivalence relations

For equivalence relations, some paths are ambiguous and some are not: eg. $|1\,3|2| \to |1|2|3| \to |1|2|3|$ is ambiguous, but $|1|2|3| \to |1\,3|2|$ is not.



Theorem: An avoidance set C of equivalence relations is wqo if and only if Γ_C contains no in-out cycles or ambiguous cycles.
(VI & N. Ruškuc, 2024)

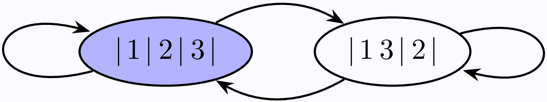
Proof outline - special vertices

Theorem

An avoidance set C of equivalence relations is wqo if and only if Γ_C contains no in-out cycles or ambiguous cycles.

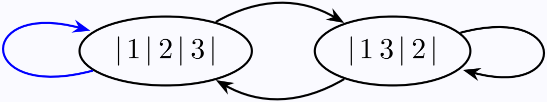
A vertex in Γ_C is special if its largest entry is in a class of size one.

So in our example, $|\,1\,|\,2\,|\,3\,|$ is a special vertex:



Proof outline - special vertices

At special vertices, new entries can be added to an existing class or a brand new class.



Consider the path $|1|2|3| \to |1|2|3|$. When the second vertex is entered, we can add the new entry either to the same class as 1 or to a new class, giving $|1|2|3|4|$ and $|1\,4|2|3|$.

Hence special vertices can lead to ambiguous paths.

Lemma

Ambiguous cycles are exactly those which contain special vertices.

Proof outline - reverse direction

Theorem

An avoidance set C of equivalence relations is wqo if and only if Γ_C contains no in-out cycles or ambiguous cycles.

Lemma

If there are no ambiguous cycles (i.e. no special vertices in cycles), there is a bound on the number of classes of equivalence relations.

This meant that we could colour the equivalence classes, creating coloured equivalence relations.

We could then identify which class is added to at each special vertex.

Proof outline - decidability

Theorem

An avoidance set C of equivalence relations is wqo if and only if Γ_C contains no ***in-out cycles*** *or* ***ambiguous cycles****.*

Since we can check for ambiguous cycles by checking for special vertices in cycles, we have the following:

Theorem: The wqo problem is decidable for equivalence relations under the consecutive order. (VI & N. Ruškuc, 2024)

Comparisons

Structure	Conditions on Γ_C for wqo
Words (McDevitt & Ruškuc, 2021)	no in-out cycles
Equivalence relations	no in-out cycles no ambiguous cycles
Permutations (McDevitt & Ruškuc, 2021) Permutations with equivalence relations (ongoing)	no in-out cycles no ambiguous cycles no splittable pairs
Graphs Digraphs Tournaments n-ary relations n binary relations	no (ambiguous) cycles

Future directions

- Investigating the wqo problem for other structures for which some paths are ambiguous and others are not.
- For some posets we cannot use factor graphs so easily, eg. for trees/forests. What happens in these cases?
- Is there an overarching picture for consecutive orders behind these results?
- Moving away from consecutive orders, can we answer the wqo problem for vincular permutation patterns?
- The wqo problem for permutations under the non-consecutive (classical) order is also open.

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Questions

Thank you for listening!