k-Hyperopic Cops and Robber

Vesna Iršič Chenoweth
Joint work with Nicholas Crawford

Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

9 April 2025

Cops and Robber Game

- Played on a simple graph G.
- Cops and a robber take turns moving along edges (or staying on the same vertex).
- The cops win if they occupy the same vertex as the robber.
- The robber wins if they evade capture indefinitely.

The smallest number of cops that ensure the win of the cops on G is the *cop number* $c(G)$ of G.

Historical Background

- Introduced independently by Quilliot (1978) and Nowakowski & Winkler (1983) (version with one cop).
- Aigner and Fromme (1984) extended it to multiple cops and defined the cop number.
- The most famous open problems in the area:
  - Meyniel's Conjecture (1987): $c(G) = O(\sqrt{n})$ for connected graphs.
  - Schröder's Conjecture (2001): $c(G) \leq g + 3$ for genus g graphs.

Variants of the Game

Robber's position is not always completely known to the cops.

Robber is *omniscient*: knows the complete strategy of the cops.

Variants of the Game

Robber's position is not always completely known to the cops.

Robber is *omniscient*: knows the complete strategy of the cops.

- 0-visibility Cops and Robber, $c_0(G)$: robber is invisible unless one of the cops is on the same vertex.
    1. introduced by Tošić (1986)
    2. known for K_n, $K_{m,n}$, P_n, C_n
    3. characterization for trees
    4. path-width is an upper bound

Variants of the Game

Robber's position is not always completely known to the cops.

Robber is *omniscient*: knows the complete strategy of the cops.

- 0-visibility Cops and Robber, $c_0(G)$: robber is invisible unless one of the cops is on the same vertex.
  1. introduced by Tošić (1986)
  2. known for K_n, $K_{m,n}$, P_n, C_n
  3. characterization for trees
  4. path-width is an upper bound
- k-visibility Cops and Robber, $c_k(G)$, $k \geq 0$: robber is invisible unless one of the cops is at distance at most k from the robber.
  1. introduced by Tang (2001) and Clarke et al. (2020)
  2. known for K_n, $K_{m,n}$, P_n, C_n
  3. characterization for trees

Variants of the Game

- hyperopic Cops and Robber, $c_H(G)$: robber is invisible if all cops are adjacent to the robber (cops are hyperopic or farsighted)
    1. introduced by Bonato et al. (2019)
    2. emulating certain prey-predator systems, where the prey has short range anti-predatory defense (e.g. a squid releasing a colored ink)
    3. inverse of the 1-visibility Cops and Robber
    4. upper bound for (outer)planar graphs
    5. Cartesian products
    6. G connected: $c_H(G) = 1$ if and only if G is a tree

New Variant

k-**hyperopic Cops and Robber:**

- cops visible
- robber is invisible if all cops are at distance between 1 and k from the robber, i.e. if r is the position of the robber, then robber is invisible if $1 \leq d(r, c) \leq k$ for every cop c
- robber is omniscient
- the minimum number of cops needed to win the game on a graph G is the *k-hyperopic cop number* $c_{H,k}(G)$ of G
- $c_{H,1}(G) = c_H(G)$
- only consider **connected** graphs

Example

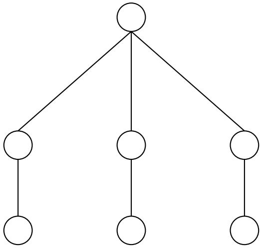

Figure: The tree $\widehat{T}$.

$c_H(\widehat{T}) = 1,\ c_{H,2}(\widehat{T}) = 2$

Basic Properties

Observation

If G is a graph and $\ell \geq k \geq 1$, then

$$c(G) \leq c_{H,k}(G) \leq c_{H,\ell}(G) \leq c_0(G).$$

Basic Properties

Observation

If G is a graph and $\ell \geq k \geq 1$, then

$$c(G) \leq c_{H,k}(G) \leq c_{H,\ell}(G) \leq c_0(G).$$

Observation

If $k \geq \text{diam}(G)$, then $c_{H,k}(G) = c_0(G)$.

More Examples

Proposition

- If $n \geq 1$, then $c_{H,k}(P_n) = 1$.
- If $n \geq 3$, then $c_{H,k}(C_n) = 2$.

More Examples

Proposition

- If $n \geq 1$, then $c_{H,k}(K_n) = \lceil \frac{n}{2} \rceil$.
- If $k \geq 2$ and $n \geq m \geq 1$, then $c_{H,k}(K_{m,n}) = m$.

“Small” k

Theorem

If $diam(G) \geq 2k+1$*, then* $c_{H,k}(G) \leq c(G)+2$*.*

“Small” k

Theorem

If $diam(G) \geq 2k+1$*, then* $c_{H,k}(G) \leq c(G)+2$*.*

The bound is sharp: $\text{diam}(G_k) = 2k+1$, $c(G_k) = 1$ and $c_{H,k}(G_k) = 3$.

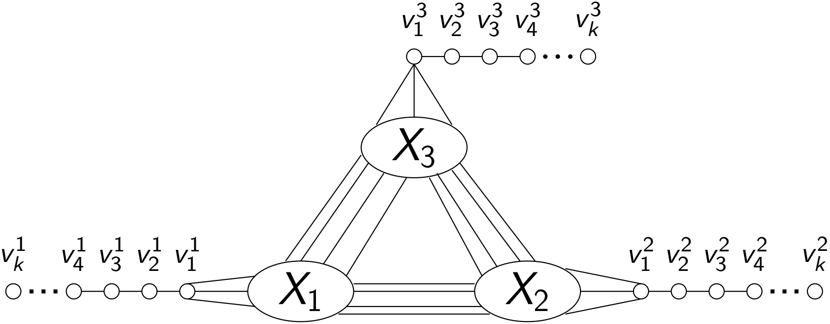

Figure: A schematic drawing of the graph G_k.

Useful Properties

Theorem

If H is a retract of G, then $c_{H,k}(H) \leq c_{H,k}(G)$.

Useful Properties

Theorem

If H is a retract of G, then $c_{H,k}(H) \leq c_{H,k}(G)$.

Recall that $c_{H,1}(G) = 1$ if and only if G is a tree.

Theorem

If $k \geq 2$ and G is a graph, then $c_{H,k}(G) = 1$ if and only if G is a caterpillar.

General Upper Bound

Recall that $c_H(G) \leq \lceil \frac{n}{2} \rceil$ for all graphs on n vertices.

General Upper Bound

Recall that $c_H(G) \leq \lceil \frac{n}{2} \rceil$ for all graphs on n vertices.

Theorem

If G is a graph, then

$$c_0(G) \leq \begin{cases} \alpha'(G), & G \text{ has a perfect matching;} \\ \alpha'(G) + 1, & \text{otherwise.} \end{cases}$$

General Upper Bound

Corollary

If G is a graph and $k \geq 1$, then

$$c_{H,k}(G) \leq \begin{cases} \alpha'(G), & G \text{ has a perfect matching;} \\ \alpha'(G) + 1, & \text{otherwise.} \end{cases}$$

Moreover, if G has n vertices, then $c_{H,k}(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

The bound is sharp: K_n and $K_m - e$ if m is even are equality cases.

0-Visibility Cops and Robber on Trees

The family $\mathcal{T}_1 = \{K_1\}$. For $m \geq 1$, the family $\mathcal{T}_{m+1}$ consists of all trees T derived from T_1, T_2, $T_2 \in \mathcal{T}_m$ (not necessarily distinct) by adding a disjoint copy of $K_{1,3}$ with vertices x, y_1, y_2, y_3 (where x is the center) and making y_i adjacent to one vertex in T_i, $i \in [3]$.

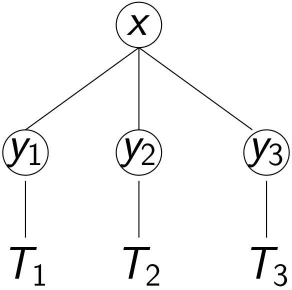

Figure: A construction of a tree in $\mathcal{T}_{m+1}$ from T_1, T_2, $T_3 \in \mathcal{T}_m$.

0-Visibility Cops and Robber on Trees

Theorem (Dereniowski, Dyer, Tifenbach, Yang (2015))

If T is a tree, then $c_0(T) \geq m$ if and only if there is $T' \in \mathcal{T}_m$ such that T' is a minor of T.

0-Visibility Cops and Robber on Trees

Theorem (Dereniowski, Dyer, Tifenbach, Yang (2015))

If T is a tree, then $c_0(T) \geq m$ if and only if there is $T' \in \mathcal{T}_m$ such that T' is a minor of T.

So there are trees with arbitrarily large k-hyperopic number!

Results on Trees

Recall that if T is a tree, then $c_{H,1}(T) = 1$.

Corollary

If T is a tree, $\text{diam}(T) \geq 4$ *and* $k \geq 1$*, then*

$$c_{H,k}(T) \leq \left\lfloor \frac{\text{diam}(T)}{4} \right\rfloor.$$

Note that if $\text{diam}(T) \leq 3$, then $c_{H,k}(T) = 1$.

Results on Trees

Recall that if T is a tree, then $c_{H,1}(T) = 1$.

Corollary

If T is a tree, $\mathrm{diam}(T) \geq 4$ *and* $k \geq 1$*, then*

$$c_{H,k}(T) \leq \left\lfloor \frac{\mathrm{diam}(T)}{4} \right\rfloor.$$

Note that if $\mathrm{diam}(T) \leq 3$, then $c_{H,k}(T) = 1$.

Theorem

If T is a tree, then $c_{H,2}(T) \leq 2$.

Proof idea.

$c_{H,2}(T) \leq 2$ for a tree T

Results on Trees

Fix k. Then we know the following:

1. $\operatorname{diam}(T) \leq k \implies c_{H,k}(T) = c_0(T)$

Results on Trees

Fix k. Then we know the following:

1. $\text{diam}(T) \leq k \implies c_{H,k}(T) = c_0(T)$
2. $k+1 \leq \text{diam}(T) \leq 2k-4 \implies c_{H,k}(T) \leq 2 + \left\lfloor \frac{k}{2} - \frac{\text{diam}(T)}{4} \right\rfloor$

Results on Trees

Fix k. Then we know the following:

1. $\mathrm{diam}(T) \leq k \implies c_{H,k}(T) = c_0(T)$
2. $k+1 \leq \mathrm{diam}(T) \leq 2k-4 \implies c_{H,k}(T) \leq 2 + \left\lfloor \frac{k}{2} - \frac{\mathrm{diam}(T)}{4} \right\rfloor$
3. $2k-3 \leq \mathrm{diam}(T) \leq 2k-2 \implies c_{H,k}(T) \leq 3$

Results on Trees

Fix k. Then we know the following:

1. $\mathrm{diam}(T) \leq k \implies c_{H,k}(T) = c_0(T)$
2. $k + 1 \leq \mathrm{diam}(T) \leq 2k - 4 \implies c_{H,k}(T) \leq 2 + \left\lfloor \frac{k}{2} - \frac{\mathrm{diam}(T)}{4} \right\rfloor$
3. $2k - 3 \leq \mathrm{diam}(T) \leq 2k - 2 \implies c_{H,k}(T) \leq 3$
4. $2k - 1 \leq \mathrm{diam}(T) \implies c_{H,k}(T) \leq 2$

Outerplanar Graphs

Recall that $c_H(G) \leq 2$ for every outerplanar graph G.

Theorem

If G is an outerplanar graph, then

$$c_{H,2}(G) \leq 2.$$

Outerplanar Graphs

Recall that $c_H(G) \leq 2$ for every outerplanar graph G.

Theorem

If G is an outerplanar graph, then

$$c_{H,2}(G) \leq 2.$$

Proposition

If G is a 2-connected outerplanar graph on $n \geq 5$ vertices, then

$$c_{H,k}(G) \leq \sqrt{2n}.$$

Open Questions

1. More on trees?

Open Questions

1. More on trees?
2. Characterize graphs G with $c_{H,k}(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$.

Open Questions

1. More on trees?
2. Characterize graphs G with $c_{H,k}(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$.
3. Does there exist a function $f(k)$ depending only on k such that for every outerplanar graph G and for every $k \geq 2$, $c_{H,k}(G) \leq f(k)$?

Open Questions

1. More on trees?
2. Characterize graphs G with $c_{H,k}(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$.
3. Does there exist a function $f(k)$ depending only on k such that for every outerplanar graph G and for every $k \geq 2$, $c_{H,k}(G) \leq f(k)$?
4. Similar for planar graphs?

Open Questions

1. More on trees?
2. Characterize graphs G with $c_{H,k}(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$.
3. Does there exist a function $f(k)$ depending only on k such that for every outerplanar graph G and for every $k \geq 2$, $c_{H,k}(G) \leq f(k)$?
4. Similar for planar graphs?
5. How does tree-width (path-width) relate to the k-hyperopic cop number?

Open Questions

1. More on trees?
2. Characterize graphs G with $c_{H,k}(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$.
3. Does there exist a function $f(k)$ depending only on k such that for every outerplanar graph G and for every $k \geq 2$, $c_{H,k}(G) \leq f(k)$?
4. Similar for planar graphs?
5. How does tree-width (path-width) relate to the k-hyperopic cop number?

Thank you!