Extremal Edge-Girth-Regular Graphs

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S. Filipovski
T. Raiman

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Girth Cycles Passing through an Edge in a Moore Graph

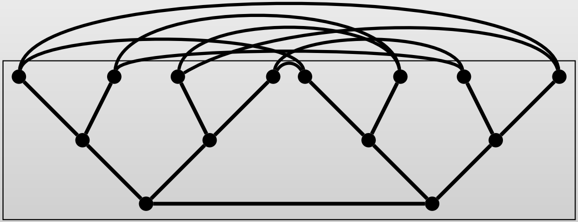

Figure: The $(3, 6)$-cage, **Heawood graph**.

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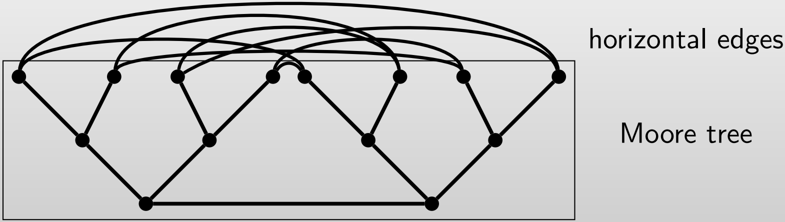

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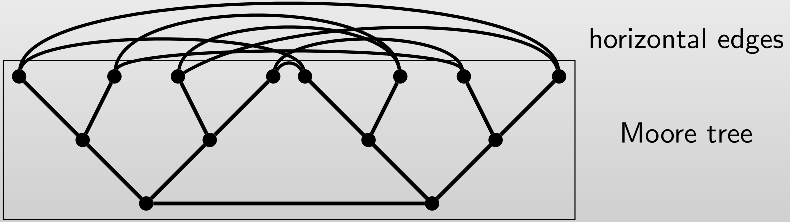

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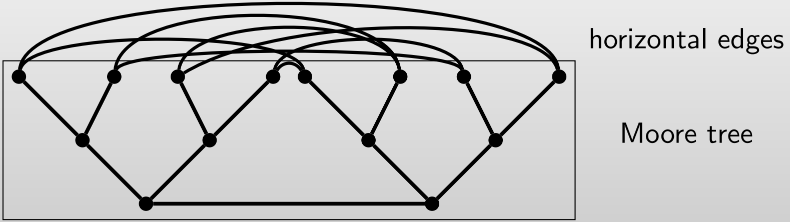

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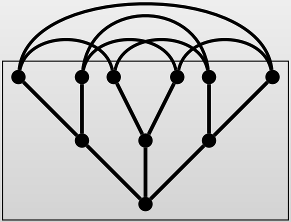
- Heawood graph is edge transitive - but is that the reason?

Girth Cycles Passing through a Vertex in a Moore Graph



horizontal edges

Moore tree

Figure: The $(3, 5)$-cage, **Petersen graph**

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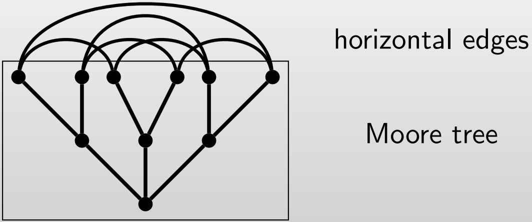

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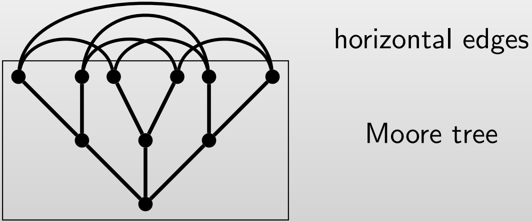

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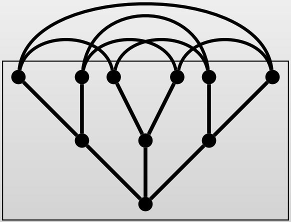

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- **each vertex and each edge sit on the same number of girth-cycles**
- this property does not seem to be the consequence of edge-transitivity
- the 'not-yet-decided' $(57,5)$-Moore cage would have to have this property, but it is known that it would be neither vertex- nor edge-transitive

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- all Moore graphs have the property that each edge sits on the same number of girth cycles
- it is not known whether every vertex (edge) of a cage must lie on a girth cycle but it sure looks desirable

Graphs Regular with Respect to Cycles

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Andries E. Brouwer, Arjeh M. Cohen, Arnold Neumaier *Distance-Regular Graphs*

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4. **distance regular graphs**: for any pair of vertices u, v of distance i, there are c_i neighbors of v in $\Gamma_{i-1}(u)$ and b_i neighbors of v in $\Gamma_{i+1}(u)$

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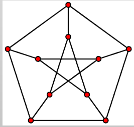 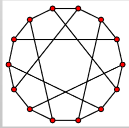 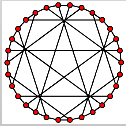

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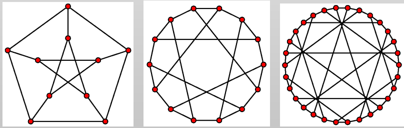

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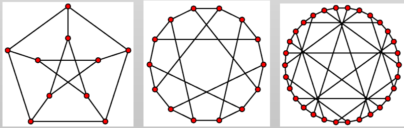

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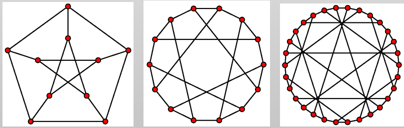

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- edge-transitive graphs have this property

Basic Observations

For all edge-girth-regular graphs

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- $$|E(G)| \cdot \lambda = (\# \text{ of } g \text{ cycles }) \cdot g$$
- $k \cdot \lambda$ must be even $\Rightarrow$ if k is odd, then λ must be even
- $$0 < \lambda \leq (k-1)^{\frac{g-1}{2}}, \text{ for odd } g$$
  $$0 < \lambda \leq (k-1)^{\frac{g}{2}}, \text{ for even } g$$

$\Rightarrow$ **Moore graphs are the natural extremal (smallest) graphs for given k and g, and if no (k,g)-Moore graph exists, λ must be smaller than the above bound**

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examples with $\lambda = 2$ often come from map theory

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  - $\lambda = 2$, **infinitely many connected graphs**, $C(\mathbb{Z}_m \times \mathbb{Z}_n, \{(1,0),(0,1),(m-1,0),(0,n-1)\})$, $m, n > 4$
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Recursive Constructions

Theorem (RJ, Kiss, Miklavič, 2018)

If there exists a (k, g, λ)-edge-girth-regular graph, there exist **infinitely many connected** $(2k, g, \lambda)$*-edge-girth-regular graphs.*

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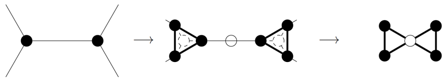

Figure 4.1: Generalized median of two vertices of order 3

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- the topic of tetravalent edge-girth-regular graphs with $\lambda = 1$ is also loosely connected to the study of **half-arc-transitive graphs**, i.e., graphs whose full automorphism group acts transitively on vertices and edges but not on vertices
- the class of **loosely G-attached** tetravalent half-arc-transitive graphs (introduced by Marušič in 1998) consists of graphs in which special alternating cycles cover all edges but any two of them share at most one vertex

Extremal Edge-Girth-Regular Graphs

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- the problem of determining the order $n(k, g, \lambda)$ for various parameter triples (k, g, λ) appears to be closely related to the *Cage Problem*
- the extremal edge-girth-regular graphs are often (but not always) edge-transitive

Lower Bounds

Theorem (Drglin, Filipovski, RJ, Raiman, 2019+)

Let k and g and be a fixed pair of integers greater than or equal to 3, and let $\lambda \leq (k-1)^{\frac{g-1}{2}}$, when g is odd, and $\lambda \leq (k-1)^{\frac{g}{2}}$, when g is even. Then

$$n(k,g,\lambda) \geq M(k,g) + (k-1)^{\frac{g-1}{2}} - \lambda, \quad \text{for } g \text{ odd}, \tag{1}$$

and

$$n(k,g,\lambda) \geq M(k,g) + \lceil 2\frac{(k-1)^{\frac{g}{2}} - \lambda}{k} \rceil, \quad \text{for } g \text{ even}. \tag{2}$$

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- odd degree regular graphs must be of even order $\Longrightarrow n(3,6,4) \geq 18$
- $\Longrightarrow$ the arc-transitive Pappus graph of order 18 is a smallest cubic edge-girth-regular graph of girth 6 and $\lambda = 4$;

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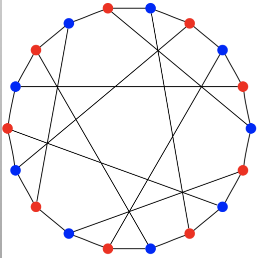

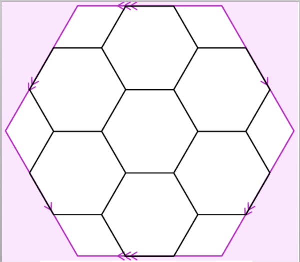

Theorem (Drglin, Filipovski, RJ, Raiman, 2019+)

Let $k \geq 3$, $g \geq 3$, and let Γ be an egr$(v,k,g,1)$-graph. Then one of the following holds:

1. *If $g \in \{3,4,5,6,7\}$, then $v \geq g + g(k-2)$.*
2. *If $g \geq 8$ and $g \equiv 0 \pmod 4$, then*
$$v \geq g + g(k-2)\left(\sum_{i=0}^{\frac{g}{4}-1}(k-1)^i\right). \quad (3)$$
3. *If $g \geq 8$ and $g \equiv 1 \pmod 4$ or $g \equiv 2 \pmod 4$, then*
$$v \geq g+g(k-2)\left(\sum_{i=0}^{\lfloor\frac{g}{4}\rfloor-1}(k-1)^i\right)+\lfloor\frac{g}{2}\rfloor(k-2)(k-1)^{\lfloor\frac{g}{4}\rfloor}. \quad (4)$$
4. *If $g \geq 8$ and $g \equiv 3 \pmod 4$, then*
$$v \geq g + g(k-2)\left(\sum_{i=0}^{\lfloor\frac{g}{4}\rfloor}(k-1)^i\right). \quad (5)$$

Lower bounds for $\lambda = 1$

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$$v \geq g +$$

$$(k-2)\left[\left(\sum_{i=0}^{\lfloor \frac{g}{2} \rfloor -1} (k-1)^i\right) + 2\left(\sum_{i=0}^{\lfloor \frac{g}{2} \rfloor -2} (k-1)^i\right) + \ldots + 2\left(\sum_{i=0}^{0} (k-1)^i\right)\right]$$

Example 2

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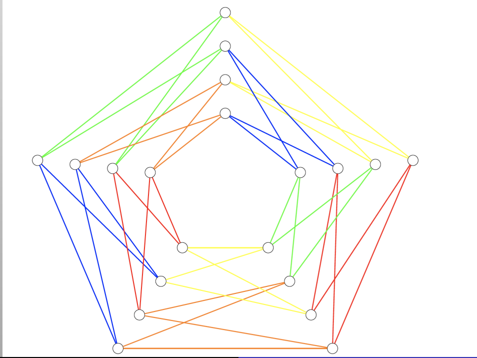

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Proof.

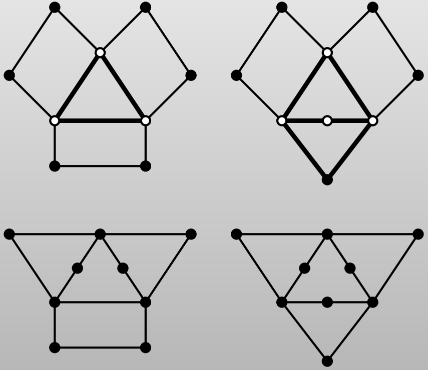

Figure: The four non-isomorphic ways three 4-cycles can pairwise share a vertex.

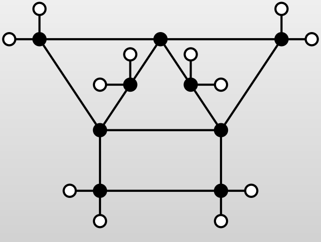

Figure: The twelve aditional vertices.

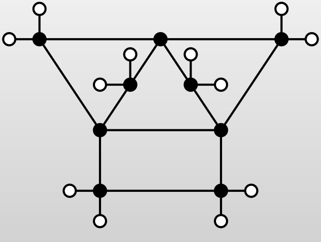
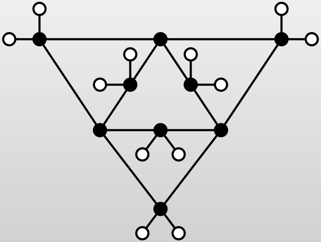

Figure: The twelve aditional vertices.

Hence, if a $egr(v, 4, 4, 1)$ with $v \leq 20$ exists, it cannot contain three 4-cycles that mutually share vertices.

Girth Cycle Graph

Definition

Let Γ be an $egr(v, k, g, \lambda)$-graph, $\lambda \geq 2$. The *girth cycle graph of* Γ, $GCG(\Gamma)$, is the graph whose vertices are the g-cycles of Γ, with two g-cycles adjacent if they share an edge.

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Let Γ be an $egr(v, k, g, 1)$-graph, and $GCG(\Gamma)$ be its associated girth cycle graph. The order of $GCG(\Gamma)$ is $\frac{vk}{2g}$, and the degree of each vertex of $GCG(\Gamma)$ is $g\frac{k-2}{2}$.

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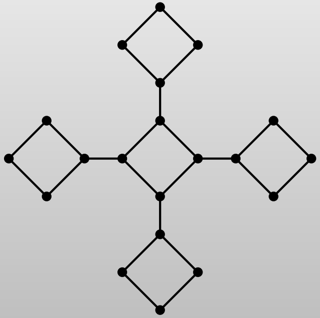
Theorem

Let Γ be an egr$(v, k, g, 1)$-graph, and GCG(Γ) be its associated girth cycle graph. The order of GCG(Γ) is $\frac{vk}{2g}$, and the degree of each vertex of GCG(Γ) is $g\frac{k-2}{2}$.

Hence, the order v of a *egr*$(v, 4, 4, 1)$ must be even, and $n(4, 4, 1) = 16, 18$ or 20.
Moreover, the *GCG* of the minimal graph cannot contain a triangle, and is therefore 4-regular of girth at least 4 and order $8, 9$ or 10.

- The only 4-regular graph of girth 4 and order 8 is $K_{4,4}$, which would mean that the minimal graph on 16 vertices would have to contain 4 independent (not touching) cycles, and hence the minimal graph would have to contain:



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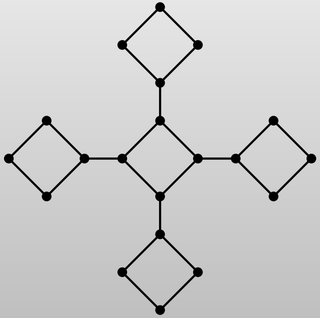

which cannot be completed into a $egr(16, 4, 4, 1)$.

- There is no 4-regular graph of girth 4 of order 9.

□

Double Cover

Theorem (Drglin, Filipovski, RJ, Raiman, 2019+)

Let Γ be a k-regular graph of odd girth g and order v. If every edge of Γ lies on λ cycles of length $g+1$, then $DC(\Gamma)$ is an $egr(2v, k, g+1, \lambda)$-graph.

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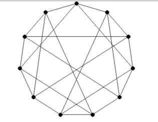
Corollary (Drglin, Filipovski, RJ, Raiman, 2019+)

For every $k \geq 3$, $n(k-1, 4, 2\binom{k-2}{2}) = 2k$.

$n(4,4,\lambda)$

- $n(4,4,9) = 8$ $K_{4,4}$
- $n(4,4,8) \geq 13$
- $n(4,4,7) \geq 14$
- $n(4,4,6) = 10$ double-cover of K_5
- $n(4,4,5) = 10$ wreath-graph of length 5, $PX(5,1)$
- $n(4,4,4) = 11$



- $n(4,4,3) = 14$ or 16 4-cube
- $12 \leq n(4,4,2) \leq 13$
- $n(4,4,1) = 20$ Praeger-Xu graph $PX(5,2)$

Open Questions

- For which situations do we have

$$n(k, g, \lambda_1) \leq n(k, g, \lambda_2), \text{ when } \lambda_1 > \lambda_2?$$

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- Is it true that the $(4, 4, 8)$-egr and $(4, 4, 7)$-egr graphs do not exist? Can we generalize this observation?
- What is the relation between cages and edge-girth-regular graphs?

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- A vertex-transitive graph does not have to be edge-girth-regular.

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- Together with Christian Rubio-Montiel and Slobodan Filipovski, we suggest to study **vertex-girth-regular graphs** and apply the results to Cayley graphs.

Girth-Regular Graphs

Main Question:

Given integers k and g, for which tuples $(a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k$ does a girth-regular graph of girth g and signature $(a_1, a_2, \ldots, a_k)$ exist?

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Theorem (Potočnik, Vidali, 2019)

If G is a connected girth-regular graph of valence k, girth $2d$ for some integer d, and signature $(a_1, a_2, \ldots, a_k)$ such that $a_k = (k-1)^d$, then $a_1 = a_2 = \ldots = a_k$ and G is the incidence graph of a generalised d-gon of order $(k-1; k-1)$.
In particular, if $k = 3$, then $g \in \{4, 6, 8, 12\}$ g and G is isomorphic to $K_{3,3}$ (if $g = 4$), the Heawood graph (if $g = 6$), the Tutte-Coxeter graph (if $g = 8$) or to the Tutte 12-cage (if $g = 12$).

Thank you.