Representations of groups and inverse semigroups on (uniform) hypergraphs

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Hypergraphs

Definition

- A **combinatorial structure** (or a **hypergraph**) $\mathcal{H} = (V, \mathcal{B})$ consists of a (finite) non-empty set V and a family $\mathcal{B}$ of subsets of V, $\mathcal{B} \subseteq \mathcal{P}(V)$.

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- if $\mathcal{B} \subseteq \mathcal{P}_k(V)$ (i.e, all the blocks are of size k), the hypergraph is a ***k*-uniform hypergraph** or simply a ***k*-hypergraph**
- the "usual" graph is a 2-hypergraph

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- An *automorphism* of $\mathcal{H}$ is a permutation of the elements of V that preserves the blocks of $\mathcal{H}$:

$$\psi(B) \in \mathcal{B} \text{ if and only if } B \in \mathcal{B}$$

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- The set of all automorphisms of $(V, \mathcal{F})$ together with the operation of composition forms the **group of automorphisms of $(V, \mathcal{F})$** denoted $Aut(\mathcal{H})$; a subgroup of $\mathbb{S}ym(V)$.

$Aut(\mathcal{H}) \leq \mathbb{S}ym(V)$.

Type of questions asked

Given a finite group G, is there a graph Γ such that $G \cong Aut(\Gamma)$?

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hypergraph

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Classification Problem:

Given a class of combinatorial structures, classify finite groups G with the property that there exists a structure from the considered class whose full automorphism group is isomorphic to G.

Example

Take $\mathbb{Z}_3$.

Is there a graph Γ such that $Aut(\Gamma) \cong \mathbb{Z}_3$?

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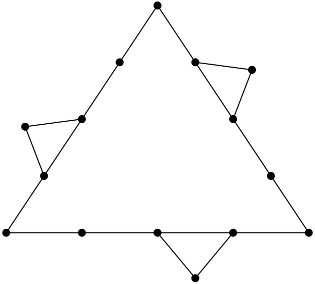

Frucht

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Theorem (Frucht 1939)

For any finite group G there exists a graph Γ such that $Aut(\Gamma) \cong G$.

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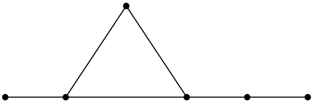
Proof.

- construct *any* $LC(G, X)$, $X = \{x_1, x_2, \ldots, x_k\}$
- find a family $X_1, X_2, \ldots, X_k$ of mutually non-isomorphic graphs that have no automorphisms (have a trivial automorphism group)
- replace each edge labeled x_i by the graph X_i, $1 \leq i \leq k$

□

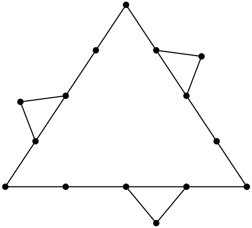


Frucht

Note: In Frucht construction, we do not specify the type of action required.

$|\mathbb{Z}_3| = 3$ $|V(\Gamma)| = 15$

Γ :



Regular Group Actions

Definition

Let G be a group acting on a set V.

- The action of G on V is said to be **regular** if for any pair of elements $u, v \in V$ there exists *exactly one* element $g \in G$ such that $u^g = v$.

Regular Group Actions

Equivalently, an action of G on V is **regular** if

- G acts transitively on V and $Stab_G(v) = 1_G$, for all $v \in V$
- G acts transitively on V and $|G| = |V|$

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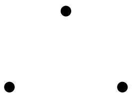

$\mathbb{S}_3$

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$\mathbb{S}_3$ $\mathbb{Z}_2$

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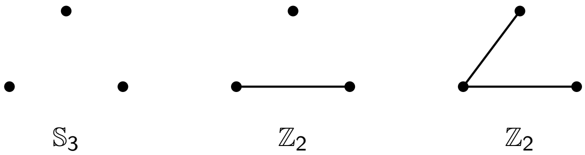

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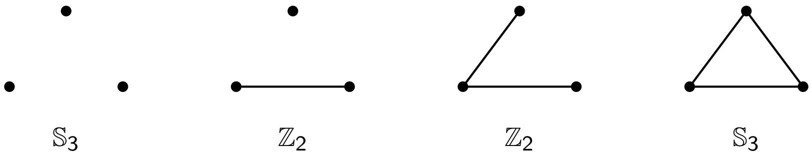

Cayley Theorem

Theorem

Every group G acts regularly on itself via (left) multiplications, i.e., G is isomorphic to the group $G_L = \{\sigma_g : G \to G \mid g \in G\}$ of (left) translations:

$$\sigma_g(h) = g \cdot h, \quad \textit{for all } h \in G$$

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Note:

- The action of G_L on G is regular.
- Every regular action of G on a set V can be viewed as the action of G_L on G.
- The orbits of σ_g on G are all of the same size $|g|$

Cayley Graphs

Definition

Given a group G, and a generating set $X = \{x_1, x_2, \dots, x_d\}$, $\langle X \rangle = G$, that is closed under taking inverses and does not contain 1_G, the vertices of the **Cayley graph** $\mathcal{C}(G, X)$ are the elements of the group G, and each vertex $g \in G$ is connected to all the vertices $gx_1, gx_2, \dots, gx_d$.

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For any $g \in G$, the **left-multiplication** $\sigma_g \in G_L$ is a graph automorphism of $\mathcal{C}(G, X)$:

$$\{a, ax\} \to \{ga, gax\}$$

for all $a \in G$ and $x \in X$,
$\Longrightarrow$

$$G_L \leq Aut(\mathcal{C}(G, X))$$

Cayley Graphs

Theorem (Sabidussi)

Let Γ *be a graph. Then* $Aut(\Gamma)$ *contains a regular group* G *if and only if* Γ *is a Cayley graph* $\mathcal{C}(G, X)$.

Cayley Graphs

Theorem (Sabidussi)

Let Γ be a graph. Then $Aut(\Gamma)$ contains a regular group G if and only if Γ is a Cayley graph $\mathcal{C}(G, X)$.

Also,

- A Cayley graph $\mathcal{C}(G, X)$ can also be viewed as the graph $\Gamma = (G, \mathcal{E})$ defined by selecting elements $X \subseteq G$, $1_G \notin X, X = X^{-1}$, of neighbors of 1_G and completing the edge set in such a way that guarantees that all permutations $\sigma_g \in G_L$ are automorphisms of Γ, i.e.,

$$\mathcal{E} = \{\sigma_g(\{1_G, x\}) \mid g \in G, x \in X\}$$

Lemma

Let $\mathcal{I} = (V, \mathcal{B})$ be a vertex transitive incidence structure. Then $\mathcal{I}$ admits a regular subgroup G of the full automorphism group $Aut(\mathcal{I})$ if and only if there exists a family of sets $B_i \in \mathcal{P}(G)$, $1 \leq i \leq k$, each of which contains 1_G, such that $\mathcal{I}$ is isomorphic to $(G, \bigcup_{i=1}^{k} B_i^{G_L})$.

Uniform Cayley Hypergraphs

Definition

Let G be a finite group, $k \geq 2$, and $X \subseteq \mathcal{P}_{k-1}(G - \{1_G\})$, the **Cayley k-hypergraph** $\mathcal{C}_k(G, X)$ is the k-hypergraph

$$(G, \bigcup_{B \in X} (B \cup \{1_G\})^{G_L})$$

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Lemma

A k-uniform hypergraph $(V, \mathcal{E})$ admits a regular group of automorphisms G if and only if it is a Cayley k-hypergraph $\mathcal{C}_k(G, X)$ for some $X \subseteq \mathcal{P}_{k-1}(G - \{1_G\})$.

Automorphism Groups of Complementary Hypergraphs

Lemma

Let $\Gamma = (V, \mathcal{H})$ *be a k-uniform hypergraph,* $1 \leq k \leq |V|$. *Then*

(i) $Aut(V, \mathcal{H}) = Aut(V, \mathcal{P}_k(V) - \mathcal{H})$;

(ii) $Aut(V, \mathcal{H}) = Aut(V, \{H^c \mid H \in \mathcal{H})$, *where* $(V, \{H^c \mid H \in \mathcal{H})$ *is the* $(|V| - k)$*-uniform hypergraph whose hyperedges are the complements of the hyperedges in* $\mathcal{H}$.

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Corollary

A finite group G has a regular representation on a k-hypergraph if and only if G has a regular representation on a $(|G| - k)$*-hypergraph.*

Other Definitions

Let G be a (finite) group, and let $X_1, X_2, \ldots, X_{k-1}$ be subsets of G that do not contain the identity 1_G.
The **C_k- hypergraph** $C_k(G; X_1, X_2, \ldots, X_{k-1})$ is the incidence structure $(G, \mathcal{H})$ with $\mathcal{H}$ being the set of all k-subsets of the form

$$\{g, gx_1, gx_1x_2, \ \ldots, \ gx_1x_2\ldots x_{k-1}\},$$

$g \in G$, and $x_i \in X_i$ for $1 \leq i \leq k-1$.

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- The C_2-hypergraph $C_2(G; X)$ is the Cayley graph $\mathcal{C}(G, X)$
- In the case when $X = X_1 = X_2 = \ldots = X_{k-1}$, the resulting blocks of $C_k(G; X, X, \ldots, X)$ are the sets of vertices corresponding to the k-arcs of the Cayley graph $\mathcal{C}(G, X)$

$$Aut(\mathcal{C}(G, X)) \leq Aut(C_k(G; X, X, \ldots, X))$$

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Lemma

Let $k \geq 2$ be an integer, and $C(G, X)$ be a Cayley graph of girth $g > 2k-2$ and valency $|X| > k-1$. Then $Aut(C(G, X)) = Aut(C_k(G; X, X, \ldots, X))$.

Other Definitions

M. Buratti, *Cayley, Marty and Schreier Hypergraphs*, 1994.

Definition

Let G be a finite group, and $S \subseteq G - \{1_G\}$. The **t-Cayley hypergraph** $t - Cay(G, S)$ has vertex set G and hyperedge set

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Since the automorphism groups of both alternative Cayley graphs contain regularly acting subgroups, both definitions are special cases of our general definition.

Regular Representations

Given a (finite) group G, find a combinatorial structure $(G, \mathcal{B})$ on G such that $Aut(G, \mathcal{B}) = G_L$.

Graphical Regular Representation

Definition

A graph Γ is a **graphical regular representation (GRR) of a group** G, if $Aut(\Gamma) = G_L$.

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Theorem (Watkins, Imrich, Godsil, ...)

Let G be a finite group that does not have a GRR, i.e., a finite group that does not admit a regular representation as the full automorphism group of a graph. Then G is an abelian group of exponent greater than 2 or G is a generalized dicyclic group or G is isomorphic to one of the 13 groups : $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_5$, $\mathcal{A}_4$, $\mathcal{Q} \times \mathbb{Z}_3$, $\mathcal{Q} \times \mathbb{Z}_4$,
$\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab \rangle$,
$\langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = ac \rangle$.

Digraphs

Theorem (Babai 1980)

The finite group G admits a DRR $\overline{C}(G,X)$ if and only if G is neither the quaternion group $\mathbb{Q}_8$ nor any of $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_3^2$.

Hypergraphical Regular Representation

The same problem as the problem of classifying finite groups admitting a GRR can also be stated for uniform hypergraphs.

Definition

A k-uniform hypergraph Γ is a **k-uniform hypergraphical regular representation (k-HRR) of a group G**, if $Aut(\Gamma) = G_L$.

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In fact, we state the problem in a more general way:

For all finite groups G, determine the spectrum of all k's for which G admits a k-uniform hypergraphical regular representation.

Groups Admitting Regular Actions on General Combinatorial Structures

Theorem

A finite group G can be represented as a regular **full automorphism group of some hypergraph** *if and only if G is not one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ or $\mathbb{Z}_2^2$.*

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The proof

- **uses blocks of different sizes**
- takes advantage of results concerning digraphs
- uses complements

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Hypergraphical Regular Representation

For all finite groups G, determine the spectrum of all k's for which G admits a k-uniform hypergraphical regular representation.

- $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ and $\mathbb{Z}_2^2$ have no k-uniform hypergraphical regular representation
- groups G that admit a GRR have a 2-uniform hypergraphical regular representation, and because of the complement lemma, they also have a $|G| - 2$-uniform hypergraphical regular representation

Hypergraphical Regular Representation

Theorem

Let $n > 5$. Then, for every k, $3 \leq k \leq n-3$, there exists a k-hypergraph $\mathcal{H}_{n,k} = (\mathbb{Z}_n, \mathcal{B})$ such that

$$Aut(\mathcal{H}_{n,k}) = \mathbb{Z}_n$$

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Proof.

Mimic DRR

$\mathcal{B} = \{\ \{i, i+1, i+2, \ldots, i+k\} \mid 0 \leq i \leq n-1\ \}$
$\cup\ \{\ \{i, i+1, i+2, \ldots, i+(k-1), i+(k+1)\} \mid 0 \leq i \leq n-1\ \}$ □

Note that cyclic groups do not admit graphical regular representation.

Diameter

Theorem

Let $\Gamma = C(G, X)$ be a Cayley graph of G of degree $k = |X|$. If Γ admits a set $\mathcal{O}$ of $2k$ vertices non-adjacent to 1_G with the property that each vertex $g \in \mathcal{O}$ belongs to a different orbit of $Stab(1_G)$, then G admits a regular representation through a 3-hypergraph.

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Corollary

Let $\Gamma = C(G, X)$ *be a Cayley graph of* G *of degree* $k = |X|$*. If* $diam(\Gamma) > 2k$*, then* G *admits a regular representation through a* 3*-hypergraph.*

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Corollary

Let $r \geq 2$. *All but finitely many finite groups of rank* r *admit regular representation through a* 3*-hypergraph.*

Girth

Lemma

Let $\Gamma = C(G, X)$ *be a Cayley graph of valency* $|X| > k - 1$ *and girth* $g > 2k - 2$, $k \geq 2$. *Then* $Aut(C(G, X)) = Aut(G, \mathcal{B})$, *where*

$$\mathcal{B} = \{ \{g, gx, gxy\} \mid g \in G, x, y \in X \}$$

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Corollary

If $\Gamma = C(G,X)$ *is a GRR for* G *of valency* $|X| > k-1$ *and girth* $g > 2k-2$, $k \geq 2$, *then* G *admits a regular representation through a 3-hypergraph.*

Computational Results

k	Groups
3	groups of *order* $= 3,4,5$ and $\mathbb{Q}_8, \mathbb{Z}_2^3$
4	groups of *order* $= 4,5,6$
5	groups of *order* $= 5,6,7$ and $\mathbb{Q}_8, \mathbb{Z}_2^3$
6	groups of *order* $= 6,7,8$
7	groups of *order* $= 7,8,9$
8	groups of *order* $= 8,9,10$
9	groups of *order* $= 9,10,11$
10	groups of *order* $= 10,11$ and $\mathbb{Z}_{12}, \mathbb{Z}_3 \rtimes \mathbb{Z}_4, \mathbb{A}_4, \mathbb{Z}_2 \times \mathbb{Z}_6$
11	groups of *order* $= 11,12,13$
12	groups of *order* $= 12,13$ and $\mathbb{Z}_{14}$
13	groups of *order* $= 13,14,15$
14	groups of *order* $= 14,15$ and $\mathbb{Z}_{16}, \mathbb{Z}_4^2, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_8 \rtimes_3 \mathbb{Z}_2, \mathbb{Q}_{16}, \mathbb{Z}_2^2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Q}_8, \mathbb{Z}_4 \circ \mathbb{D}_4, \mathbb{Z}_2^4$
15	groups of *order* $= 15,16,17$
16	groups of *order* $= 16,17$ and $\mathbb{Z}_{18}, \mathbb{Z}_3 \rtimes, \mathbb{S}_3, \mathbb{Z}_3 \times \mathbb{Z}_6$
17	groups of *order* $= 17,18,19$
18	groups of *order* $= 18,19$ and $\mathbb{Z}_{20}, \mathbb{Z}_5 \rtimes_2 \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_{10}$
19	groups of *order* $= 19,20$ and $\mathbb{Z}_{21}$
20	groups of *order* $= 20,21$ and $\mathbb{Z}_{22}$
21	groups of *order* $= 21,22,23$
22	groups of *order* $= 22,23$ and $\mathbb{Z}_{24}, \mathbb{Z}_3 \rtimes \mathbb{Q}_8, \mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4), \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_3 \times \mathbb{Q}_8, \mathbb{Z}_2^2 \times \mathbb{Z}_6$
23	groups of *order* $= 23,24,25$
24	groups of *order* $= 24,25$ and $\mathbb{Z}_{26}$
25	groups of *order* $= 25,26$ and $\mathbb{Z}_{27}, \mathbb{Z}_3 \times \mathbb{Z}_9, \mathbb{Z}_3^2 \rtimes \mathbb{Z}_3, \mathbb{Z}_3^3$
26	groups of *order* $= 26,27$ and $\mathbb{Z}_{28}, \mathbb{Z}_7 \rtimes \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_{14}$
27	groups of *order* $= 27,28,29$
28	groups of *order* $= 28,29$ and $\mathbb{Z}_{30}$
29	groups of *order* $= 29,30,31$
30	groups of *order* $= 30,31$ and $\mathbb{Z}_{32}, \mathbb{Z}_4 \times \mathbb{Z}_8, \mathbb{Z}_2.\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_{16}, \mathbb{Q}_{32}, \mathbb{Z}_2 \times \mathbb{Z}_4^2, \mathbb{Z}_2 \times \mathbb{Z}_4 \rtimes \mathbb{Z}_4, \mathbb{Z}_2 \times (\mathbb{Z}_8 \rtimes_3 \mathbb{Z}_2), \mathbb{Z}_4 \rtimes \mathbb{Q}_8, \mathbb{Z}_2^2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Q}_{16}, \mathbb{Z}_2^3 \times \mathbb{Z}_4, \mathbb{Z}_2^2 \times \mathbb{Q}_8$
31	groups of *order* $= 31,32$
32	groups of *order* $= 32$

Table: Groups of order ≤ 32 not admitting HRR's via k-hypergraphs for k in the range $0 \leq k \leq |G|$.

Conjectures

- For every finite group G, the spectrum of k's for which there exists a k-uniform hypergraphical regular representation is connected and symmetric.

Classical approach to symmetries

- For a combinatorial structure $\mathcal{C} = (V, \mathcal{B})$
- we study automorphisms of $(V, \mathcal{B})$
- Automorphisms form a group, $Aut(\mathcal{C}) \leq Sym(V)$.

Asymmetric graphs

Theorem (Erdös, Rényi, 1963)

Almost all finite graphs are asymmetric.

Asymmetric graphs

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Almost all finite graphs are asymmetric.

A graph Γ is called *asymmetric* if it does not have a non-trivial automorphis, so its automorphisms group $Aut(\Gamma)$ is trivial.

Asymmetric graphs

A graph G is called *asymmetric* if it does not have a non-trivial automorphism.

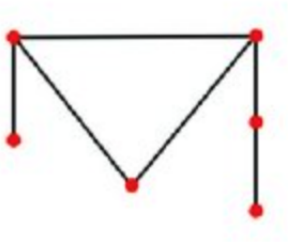


Figure: The smallest asymmetric graph

Asymmetric graphs

A graph G is called *asymmetric* if it does not have a non-trivial automorphism.

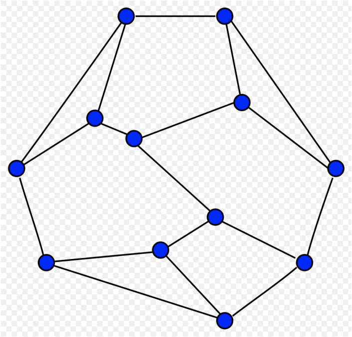


Figure: The Frucht graph, one of the five smallest asymmetric cubic graphs.

Measure of asymmetricity

How many vertices of a graph do we need to remove to get a symmetric graph?

Minimal asymmetric graphs

An undirected graph G on at least two vertices is *minimal asymmetric* if G is asymmetric and no proper induced subgraph of G on at least two vertices is asymmetric.

Minimal asymmetric graphs

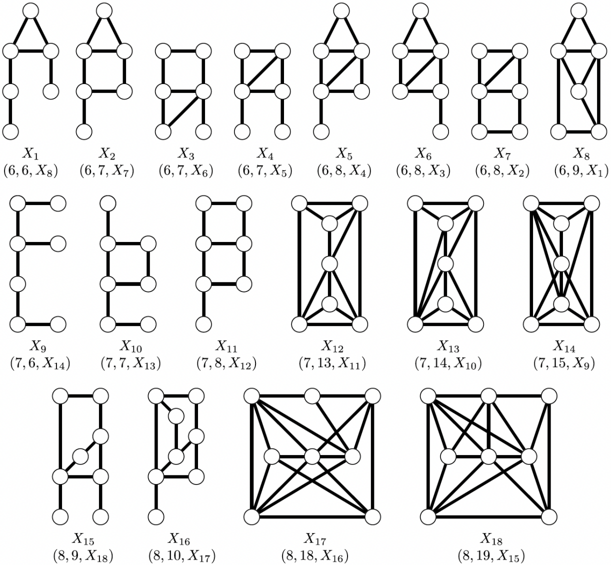
Theorem (Schweitzer, Pascal; Schweitzer, Patrick, 2017)

There are exactly 18 finite minimal asymmetric undirected graphs up to isomorphism.

Nešetřil's conjecture: There are exactly 18 minimal asymmetric graphs (coming in 9 complementary pairs).

Nešetřil and Sabidussi earlier established a close connection between minimal asymmetric graphs and minimal involution-free graphs.

Minimal asymmetric graphs



Partial graph automorphisms

Let $\Gamma = (V, \mathcal{E})$ be a finite graph

A partial automorphism of $\Gamma = (V, \mathcal{E})$ is an isomorphism between two *induced* subgraphs.

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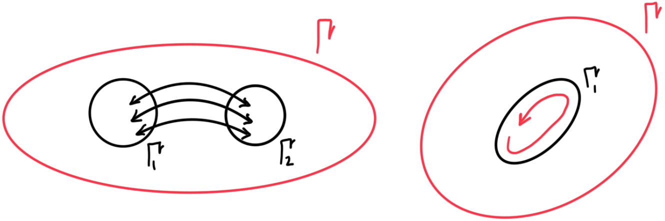

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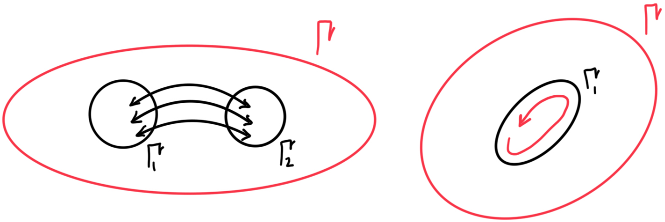

The set of all partial automorphisms, denoted $\mathrm{PAut}(\Gamma)$ with the composition and partial inverse of partial maps forms an inverse monoid.

$\mathrm{PAut}(\Gamma) \leq \mathrm{PSym}(V)$

Inverse Monoid

A set together with an associative binary operation is called a *semigroup*

A semigroup having an identity element is a *monoid*

Inverse Monoid

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A semigroup having an identity element is a *monoid*

A monoid M is called **inverse**

- if for every $a \in M$ there exists a unique element a^{-1} s.t.

$$a \cdot a^{-1} \cdot a = a$$

$$a^{-1} \cdot a \cdot a^{-1} = a^{-1}$$

"Archetypal" inverse semigroup PSym(X)

PSym(X) - set of all partial permutations of X
$=$ bijections between subsets of X (including $\emptyset$).

$$\varphi : Y \to Z \qquad Y, Z \subseteq X$$

Y - domain $\quad dom\varphi$
Z - range $\quad ran\varphi$

$|dom\varphi| = |ran\varphi|$ - rank of φ

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$|dom\varphi| = |ran\varphi|$ - rank of φ

The cycle notation of classical permutations generalizes by the addition of a notion called a path, which (unlike a cycle) ends when it reaches the "undefined" element.

$$\text{dom}(x_1, x_2 ... x_k] = \{x_1, x_2, ..., x_{k-1}\}$$

$$\text{ran}(x_1, x_2 ... x_k] = \{x_2, x_3, ..., x_k\}$$

PSym(X)

- Operation on PSym(X) - is composition of partial maps:
  Let α and β be partial permutations of a set X; α and β can be composed (from left to right) on the largest domain upon which it "makes sense" to compose them (may be the $\emptyset$)

$$\text{dom}\, \alpha\beta = [\text{im}\, \alpha \cap \text{dom}\, \beta]\alpha^{-1}$$

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- Identity of PSym(X) - is id_X
- Local Identities: $A \subset X$ - id_A, idempotents
- Zero - PSym(X) has also zero element - empty map - $id_\emptyset$

Wagner-Preston representation

While groups can be represented as symmetries:

Theorem (Cayley)

Every (finite) group can be represented as a group of permutations of a (finite) set.

Inverse semigroups can be represented as partial symmetries:

Theorem (Wagner-Preston)

Every (finite) inverse semigroup can be represented as the inverse semigroup of partial bijections of a (finite) set.

Restrictions vs. extensions

It is clear that all restrictions of an automorphism of a graph are partial automorphisms.

But not all partial automorphisms extend to a (global) automorphism.

Homogeneous graphs

A graph Γ is *homogeneous* if any isomorphism between induced subgraphs extends to an automorphism of Γ.

Theorem (Gardiner 1976)

A finite homogeneous graph is one of the following:

- *a disjoint union of complete graphs of the same size*
- *a regular complete multipartite graph*
- *the 5-cycle*
- *the line graph of $K_{3,3}$*

Hrushovski's extension theorem

Theorem (Hrushovski 1992)

Let X be a finite graph. Then there exists a finite graph Z containing X as an induced subgraph, such that every isomorphism between induced subgraphs of X extends to an automorphism of Z.

Extensions of partial automorphisms

Jaroslav Nešetřil, Matěj Konečný, ...
Extension property for partial automorphisms, EPPA

Let A be a structure and let B be its (induced) substructure. A is an EPPA-witness for B if every partial automorphism of B extends to an automorphism of A.

A class $\mathcal{C}$ of **finite** structures has EPPA if for every $B \in \mathcal{C}$ there is $A \in \mathcal{C}$, which is an EPPA-witness for B.

Theorem (Hrushovski 1992)

The class of all finite graphs has EPPA.

Extensions of partial automorphisms

Jaroslav Nešetřil, Matěj Konečný, ...
Extension property for partial automorphisms, EPPA

- Class of all n-partite tournaments (orientations of complete n-partite graphs) has EPPA (Eurocomb2019)
- The question is still open for the class of *all* tournaments

Reconstruction

Definition

Let $\Gamma = (V, \mathcal{E})$ be a finite graph and $u \in V(\Gamma)$.
Then $\Gamma - u$ is called a *card*.
The collection $\mathcal{D}$ of the cards of a graph Γ is called the **deck** of Γ:
$\mathcal{D}$ is the multiset of all induced subgraphs $\Gamma - u$, $u \in V$.

Graph reconstruction conjecture (Kelly and Ulam, 1957):

Every finite graph on at least 3 vertices is
uniquely reconstructible from its deck.

Pseudo-similarity

Definition

Two vertices $u, v \in V$ are pseudo-similar if $\Gamma - \{u\}$ and $\Gamma - \{v\}$ are isomorphic, but there exists no automorphism of Γ mapping u to v.

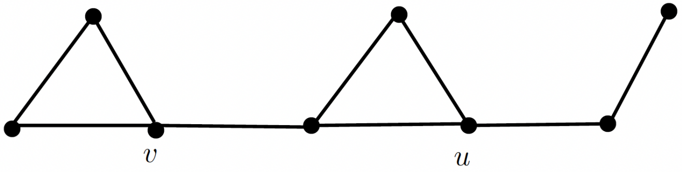


Figure: The Harary-Palmer Graph - the smallest graph containing a pair of pseudo-similar vertices

Cages

Definition

A k-regular graph Γ of girth g is called a (k,g)-cage if Γ is of smallest possible order among all k-regular graphs of girth g.

Open problem: Does there exist a $(57,5)$-graph of order 3250?

We do know that if the graph exists, it is not vertex-transitive, but for any two vertices u, v of such graph, there would exist a partial automorphism mapping u to v whose domain would constitute a significant part of the graph.

Inverse monoids of partial graph automorphisms

Main Questions:

1. Classify finite inverse monoids that are *isomorphic* to inverse monoids of partial automorphisms of a graph

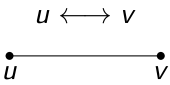
Analogue of Frucht's theorem for groups.

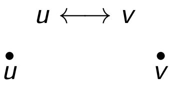
2. For a specific class of representations of finite inverse semigroups (e.g., those given by Wagner-Preston theorem) classify finite inverse semigroups that admit a combinatorial structure for which the inverse semigroup of partial automorphisms is *equal to* the partial bijections from the representation.

Analogue of GRR's for groups.

Analogue of Frucht's theorem

Note: The inverse semigroup of partial automorphisms of a *graph* $\Gamma = (V, \mathcal{E})$ with more than one vertex is never trivial:





Note: The inverse semigroup of partial automorphisms of a *graph* $\Gamma = (V, \mathcal{E})$ with more than one vertex is never trivial:

Corollary

Not every finite inverse monoid is the inverse monoid of partial automorphisms of a graph.

Classification of inverse semigroups of partial automorphisms of combinatorial structures

Theorem (Nemirovskaya 1997)

If S is a finite inverse semigroup, then there exists a **weighted graph** Γ *such that* $S \cong PAut_\omega(\Gamma)$.

Theorem (Sieben 2008)

The inverse semigroup of partial automorphisms of the **Cayley color graph** *of an inverse semigroup is isomorphic to the original inverse semigroup.*

Structure of inverse monoids

$e \in S$ is an idempotent, if $e^2 = e$.

$E(S)$ - set of all idempotents of S.

$\forall s \in S$, $ss^{-1}, s^{-1}s \in E(S)$ (generally different)

- in inverse monoids, idempotents commute
- they form a subsemilattice
- the partial order induced by this semilattice extends naturally to the whole inverse semigroup:

  $$s \leq t \Leftrightarrow \exists \text{ an idempotent } e \text{ such that } s = te$$

  This is called the *natural partial order*

Structure of inverse monoids

In $PSym(X)$,

- idempotents are the partial identical maps,
- the natural partial order is defined by restriction of domains.

PAut(Γ) of a graph Γ is a *full* submonoid (= contains all idempotents) of PSym(V).

Structure of inverse monoids

Green's relations:

$s, t \in S$. We define $\mathcal{L}$ and $\mathcal{R}$:

- $s\ \mathcal{L}\ t \Leftrightarrow \exists\ x, y \in S$ s.t. $xs = t\ \&\ yt = s$,
- $s\ \mathcal{R}\ t \Leftrightarrow \exists\ x, y \in S$ s.t. $sx = t\ \&\ ty = s$.

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In $PSym(X)$,

$$\varphi_1 \,\mathcal{L}\, \varphi_2 \Leftrightarrow \operatorname{dom} \varphi_1 = \operatorname{dom} \varphi_2,$$
$$\varphi_1 \,\mathcal{R}\, \varphi_2 \Leftrightarrow \operatorname{ran} \varphi_1 = \operatorname{ran} \varphi_2.$$

Green's relations:

$\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
$\mathcal{D} = \mathcal{R} \vee \mathcal{L}$
We can show $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$

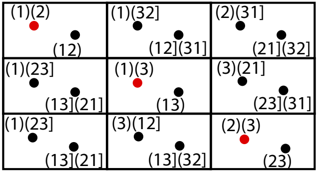

Figure: The $\mathcal{D}$-class of rank 2 partial one-to-one maps of PSym($\{1, 2, 3\}$)

Structure of PAut(Γ) for graph Γ

Proposition (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

For any graph Γ*, the* $\mathcal{D}$*-classes of* PAut(Γ) *correspond to the isomorphism classes of induced subgraphs of* Γ*, that is, two elements are* $\mathcal{D}$*-related if and only if the subgraphs induced by their respective domains (or images) are isomorphic.*

Structure of PAut(Γ) for graph Γ

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Partial order for $\mathcal{D}$-classes: "subgraph" relation

Example

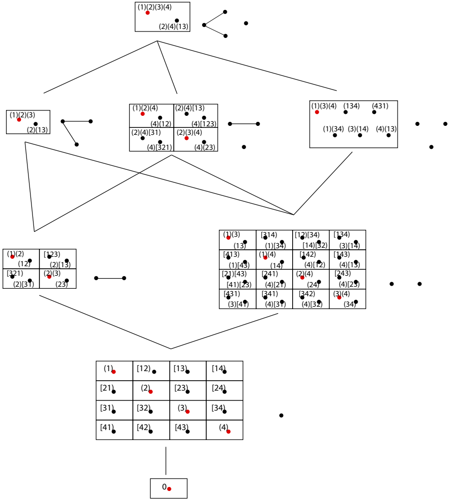

Figure: The Green-class structure of partial graph automorphisms

Structure of $\mathrm{PAut}(\Gamma)$ for graph Γ

Lemma (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

Let $\Gamma = (X, E)$ be a graph, and let $\varphi \in \mathrm{PSym}(X)$ be a partial permutation of rank at least 2. Then $\varphi \in \mathrm{PAut}(\Gamma)$ if and only if $\varphi|_Y \in \mathrm{PAut}(\Gamma)$ for any 2-element subset Y of $\mathrm{dom}\,\varphi$.

Proposition

The partial automorphism monoid $\mathcal{S} = \mathrm{PAut}(\Gamma)$ of any graph Γ has the following property:

(U) *For any compatible subset $A \subseteq \mathcal{S}$ of partial permutations of rank 1, if $\mathcal{S}$ contains the join of any two elements of A, then $\mathcal{S}$ contains the join of the set A.*

Proposition

If $\mathcal{S}, \mathcal{T}$ are full inverse submonoids of $\mathrm{PSym}(X)$ which coincide on their elements of rank at most 2 and satisfy condition (U), then $\mathcal{S} = \mathcal{T}$.

When is an inverse monoid of partial permutations the partial automorphism monoid of a graph?

Theorem (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

Given an inverse submonoid $S \leq \mathrm{PSym}(X)$, where X is a finite set, there exists a graph with vertex set X whose partial automorphism monoid is S if and only if the following conditions hold:

1. *S is a full inverse submonoid of $\mathrm{PSym}(X)$,*
2. *for any compatible subset $A \subseteq S$ of rank 1 partial permutations, if S contains the join of any two elements of A, then S contains the join of the set A,*
3. *the rank 2 elements of S form at most two $\mathcal{D}$-classes,*
4. *the $\mathcal{H}$-classes of rank 2 elements are nontrivial.*

When is an (abstract) inverse monoid *isomorphic* to the partial automorphism monoid of a graph?

Theorem (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

Given a finite inverse monoid S, there exists a finite graph whose partial automorphism monoid is isomorphic to S if and only if the following conditions hold:

1. *S is Boolean,*
2. *S is fundamental,*
3. *for any subset $A \subseteq S$ of compatible 0-minimal elements, if all 2-element subsets of A have a join in S, then the set A has a join in S,*
4. *S has at most two $\mathcal{D}$-classes of height 2,*
5. *the $\mathcal{H}$-classes of the height 2 $\mathcal{D}$-classes of S are nontrivial.*

Work in progress

- For a graph (edge-colored graph, digraph) Γ, we can construct its $\mathrm{PAut}(\Gamma)$
- We now understand the structure of $\mathrm{PAut}(\Gamma)$ for some interesting (but still quite simple) classes of graphs
  (joint work with M. Méri)
- We contructed the catalogue of $\mathrm{PAut}(\Gamma)$ for minimal asymmetric graphs
  (diploma theses of M. Gál)
- Not every finite inverse monoid is the inverse monoid of partial automorphisms of a graph.
- Plan: Characterize those inverse monoids that can be regagularly represented by a combinatorial structure (say uniform k-hypergraphs).
  (project with Dominika Mihálová)

Thank you!

PF 2023!