Genuinely edge-biregular maps exist for almost all feasible types

with the underlying colour-preserving automorphism group being alternating or symmetric

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A map with an alternate-edge-colouring

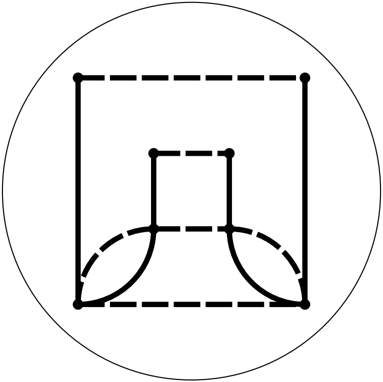


Figure: A map on a sphere with an assigned alternate-edge-colouring

An edge-biregular map

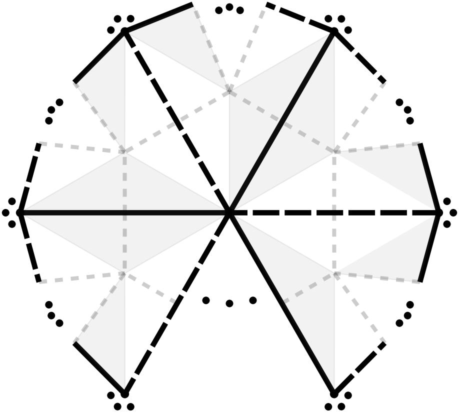


Figure: An edge-biregular map: two orbits on the flags

The colour-preserving automorphism group H acts regularly on the corners of an edge-biregular map.

An edge-biregular map and H, its colour-preserving automorphism group

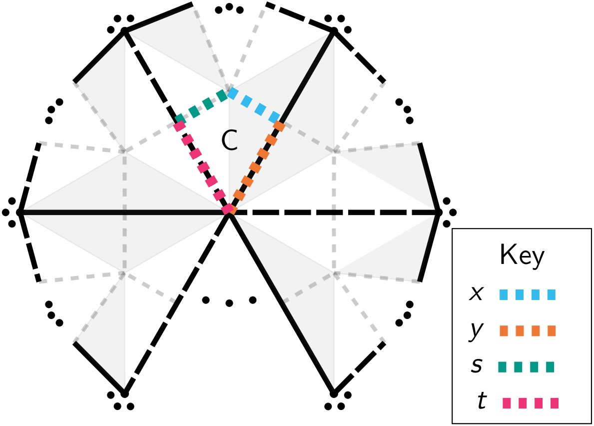


Figure: Generators of H for an edge-biregular map of type (k, l)

$$H = \langle\, x,\ y,\ s,\ t \mid x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^{k/2},\ (xs)^{l/2},\ \ldots \rangle$$

Example: an edge-biregular map of type $(8, 4)$

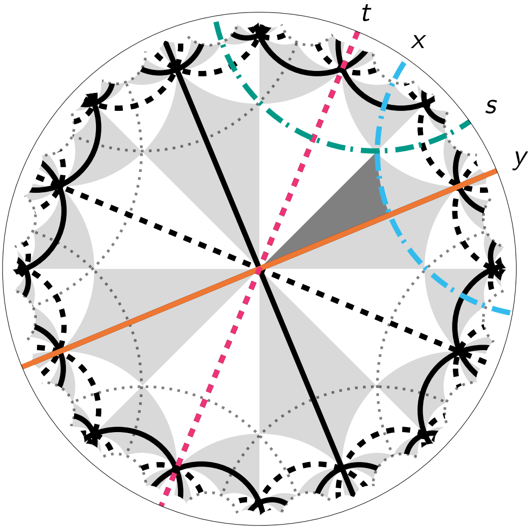


Figure: The hyperbolic lines of reflection for the generating automorphisms x, y, s and t, shown on part of the infinite edge-biregular map.

A genuinely edge-biregular map...

... is an edge-biregular map which is not fully regular.
Then H is the automorphism group for the map.

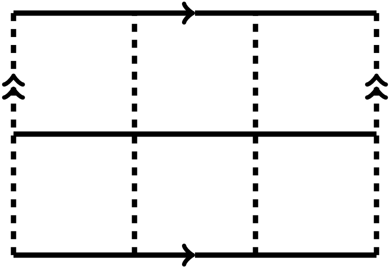


Figure: A genuinely edge-biregular map of type $(4, 4)$ on a torus

There is no automorphism of the group H which interchanges x with s and y with t:
$\langle\, x,\ y,\ s,\ t \mid x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^2,\ (xs)^2,\ (sy)^2,\ (xt)^3 \rangle$

What was the question?

Do genuinely edge-biregular maps exist for every feasible type $(2\kappa, 2\lambda)$?

Notice: We may assume $\kappa \leq \lambda$, simply by using the dual map if necessary.

Small (non-)examples:

Maps of type $(2, 2\kappa)$ and $(2\lambda, 2)$ are known to be spherical...
... and are all *fully regular* maps.

Type $(4, 4)$:
The toroidal example shows existence for genuine edge-biregularity...
... but H is not alternating nor symmetric.

From now on, we may also assume that $\kappa \geq 2$ and $\lambda \geq 3$.

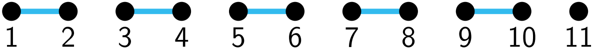
Where else could we look?

Existing classifications of edge-biregular maps (R and Širáň) :
... with H being a dihedral group;
... on surfaces of small Euler characteristic;
... on surfaces with negative prime Euler characteristic.
Not enough - the types are restricted.

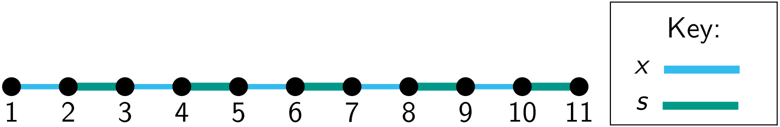
So let's start from scratch with another viewpoint:
Identify groups with the necessary properties.
$H = \langle\, x,\ y,\ s,\ t \ \mid\ x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^{\kappa},\ (xs)^{\lambda},\ \ldots \rangle$
Permuation groups - and their associated diagrams......

$H = \langle\, x,\ y,\ s,\ t \mid x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^{\kappa},\ (xs)^{\lambda},\ \dots \rangle$

Involutions are products of disjoint transpositions.



Two-involution chain:

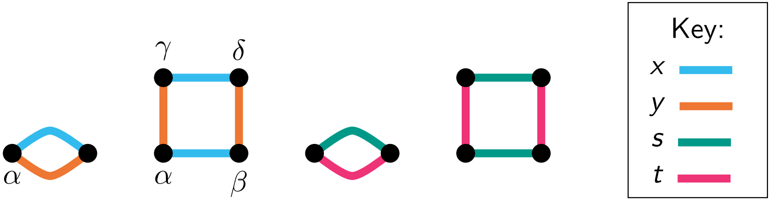


xs is a single cycle.
Notice: the order of the product is the same as the number of points in the (connected) chain.

Interaction of commuting involutions in permutation diagrams.

The group H for an edge-biregular map demands that x and y commute, that is $ord(xy) = 2$.

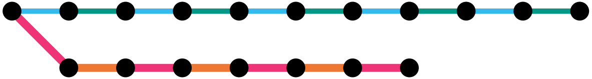
This gives only two possibilities for when a given point α is fixed by neither x nor y:
Either $\alpha x = \alpha y$ or $\alpha x = \beta \neq \gamma = \alpha y$ but then $\beta xyx = \alpha yx = \gamma x := \delta$ and so $\delta x = \gamma$ while $\delta y = \beta xyxy = \beta$.



The two involutions x and y commute. Also s and t commute.

A small example:

$H = \langle\, x,\ y,\ s,\ t \mid x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^\kappa,\ (xs)^\lambda,\ \ldots \rangle$
An *xs* chain and a *ty* chain meeting at just one point:



but
Is the map genuinely edge-biregular?
Is the construction 'efficient'?
What group is generated?
Can I minimise the degree of the premutation group?
Surely we can do better...!

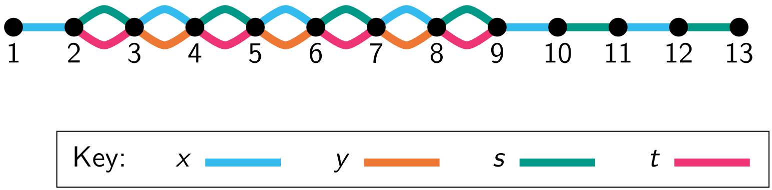
$H = \langle\, x,\ y,\ s,\ t \mid x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^\kappa,\ (xs)^\lambda,\ \dots \rangle$

What group is generated? I aim for alternating or symmetric groups...

Theorem (Jones)

Let G be a primitive permutation group of finite degree n, containing a cycle with k fixed points. Then $G \geq A_n$ if $k \geq 3$.

Example: A construction for when $6 \leq \kappa + 3 \leq \lambda$



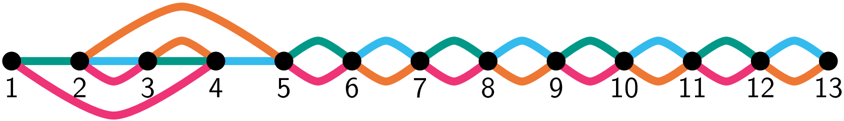
Primitive - yes: think. Short cycle - yes: yt. S_{13}

Concern:

What if κ and λ are too close to each other, or even equal?

A solution: When $6 \leq \kappa = \lambda = m$:

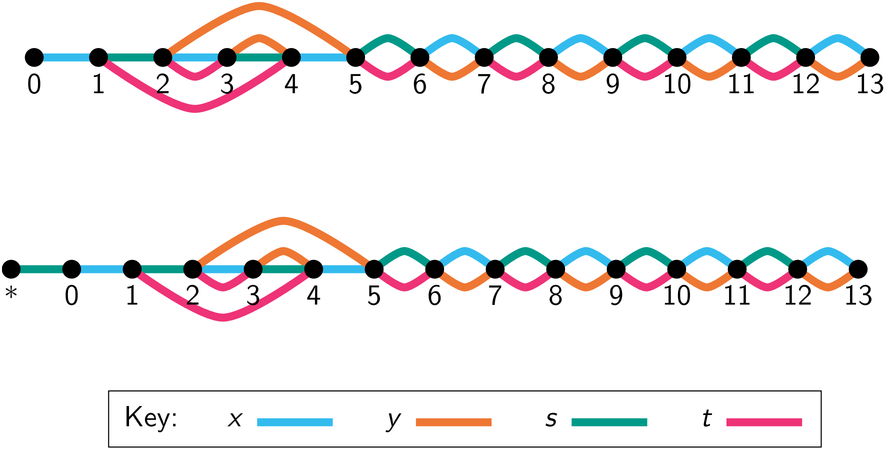
$H = \langle\, x,\ y,\ s,\ t \mid x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^{\kappa},\ (xs)^{\lambda},\ \ldots \rangle$



Key: x — y — s — t —

Proof: (panic not - it generalises)
Orders of elements are as required.
Primitive: sy fixes points 3 and 4. tx fixes points 2 and 3.
Short cycle: $xyst = (1,3,5)(2)(4)(6)(7)...$
Alternating or symmetric, of degree λ, by Jones.
$sxy = (1,4,5,6,3,2)(7,8)(9,10)(11,12)(13)$
$xst = (1,3,4,5,2)(6,7)(8,9)(10,11)(12,13)$
Genuine!

The cases in between: When $1 \leq \lambda - \kappa \leq 2$ ...



... the constructions look remarkably similar, as do the associated proofs.

Are these the 'best'?!

These alternating/symmetric constructions are extremal (in that the degree of the permutation group is minimised) when λ is prime.

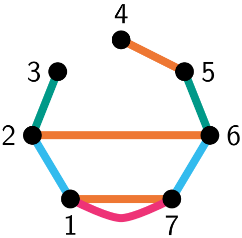
A table to cover some of the smaller extremal examples...

$$H = \langle\, x,\ y,\ s,\ t \mid x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (yt)^{k/2},\ (xs)^{l/2},\ \ldots \rangle$$

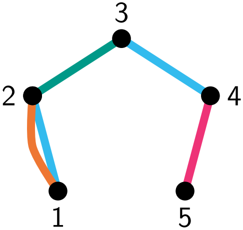
Type	x	y	s	t	$H \cong$
$(6,6)$	(12)	(34)	(23)	(45)	S_5
$(6,8)$	$(12)(34)$	(12)	(23)	(14)	S_4
$(6,10)$	$(12)(34)$	(12)	$(23)(45)$	(23)	S_5
$(8,8)$	$(12)(34)$	(34)	(23)	$(14)(23)$	S_4
$(8,10)$	$(12)(34)$	$(13)(24)$	$(23)(45)$	(23)	S_5
$(8,12)$	$(12)(34)$	$(14)(23)$	(45)	(13)	S_5
$(10,10)$	$(12)(34)$	$(13)(24)$	$(23)(45)$	$(34)(25)$	A_5
$(10,12)$	(12)	$(12)(34)$	$(25)(34)$	$(23)(45)$	S_5
$(10,14)$	$(12)(34)(56)$	$(13)(24)$	$(23)(45)(67)$	$(23)(45)$	S_7

Table: Generators for extremal examples of genuinely edge-biregular maps on alternating and symmetric groups.

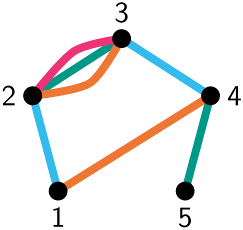
... but I like pictures!!! This covers the rest of the small examples:



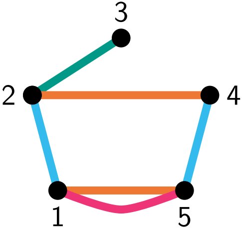
Type $(4, 6)$



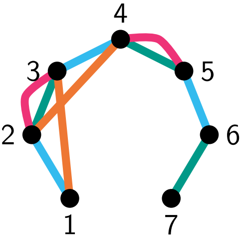
Type $(4, 8)$



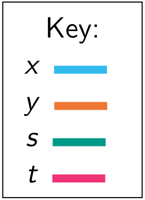
Type $(4, 10)$



Type $(4, 12)$



Type $(10, 14)$



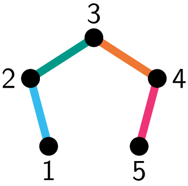
Conclusion

Theorem

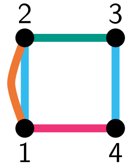
For any given pair of integers $\kappa \geq 2$ and $\lambda \geq 2$, there exists $\mathcal{M} = (H; x, y, s, t)$, a genuinely edge-biregular map of type $(2\kappa, 2\lambda)$. When $\kappa + \lambda = 4$ the group H is not isomorphic to a symmetric or alternating group, but when $\kappa + \lambda \geq 5$, the group H can be isomorphic to a symmetric group $H \cong S_m$, or an alternating group $H \cong A_m$. In particular, extremal examples having minimal degree m are as follows:

1. *When $\kappa + \lambda = 5$, the extremal examples are such that $H \cong S_7$.*
2. *When $\kappa + \lambda = 6$, the extremal examples are such that $H \cong S_5$.*
3. *When $\kappa + \lambda \geq 7$, the colour-preserving automorphism group H can be isomorphic to a symmetric or alternating group with minimal degree no more than $\max\{\kappa, \lambda\}$.*

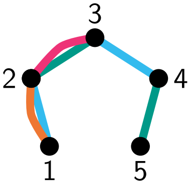
The end - thanks for being (t)here!!! :-)



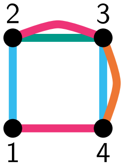
Type $(6, 6)$



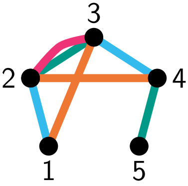
Type $(6, 8)$



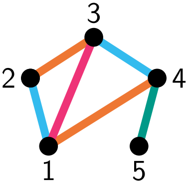
Type $(6, 10)$



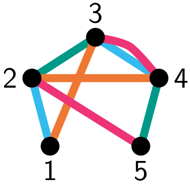
Type $(8, 8)$



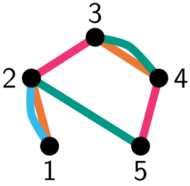
Type $(8, 10)$



Type $(8, 12)$



Type $(10, 10)$



Type $(10, 12)$