KU LEUVEN kulak fwo Research Foundation Flanders Opening new horizons

On vertex-girth-regular graphs: (non-)existence, bounds and enumeration

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0 Outline

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1 Motivating problem: the cage problem

Definition

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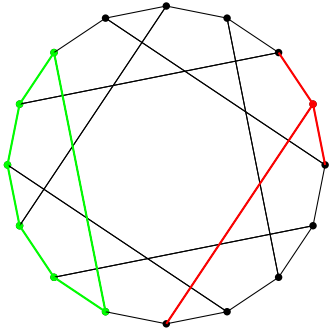
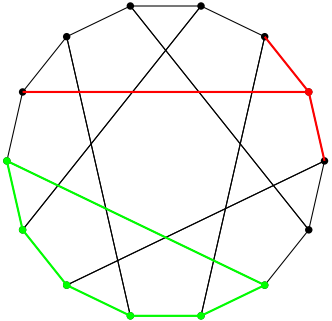
Cage problem: for given integers k and g, find the smallest v such that a (v, k, g) graph exists (denoted by $n(k, g)$) as well as the corresponding graphs, called (k, g)-*cages*.

1 Examples: Möbius-Kantor graph and Heawood graph



(a) A $(16, 3, 6)$ graph (Möbius-Kantor)



(b) The $(14, 3, 6)$ cage (Heawood).

We have $\mathbf{n(3, 6) = 14}$.

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- **How to progress?** Improve fundamental understanding by studying strongly related problems.

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k	g	$n(k, g)$	vertex-transitive	Cayley
3	6	**14**	**14**	**14**
3	7	**24**	26	30
3	8	**30**	**30**	42
3	9	**58**	60	60
3	10	**70**	80	96

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- We aim to study classes of graphs of 'just the right level of bias'

1 Vertex-girth-regular graphs

Observation: many (k, g)-cages have the property that each vertex is contained in the same number of cycles of length g (*girth cycles*).

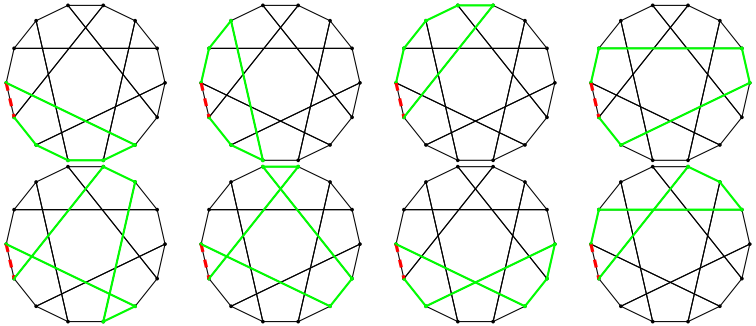
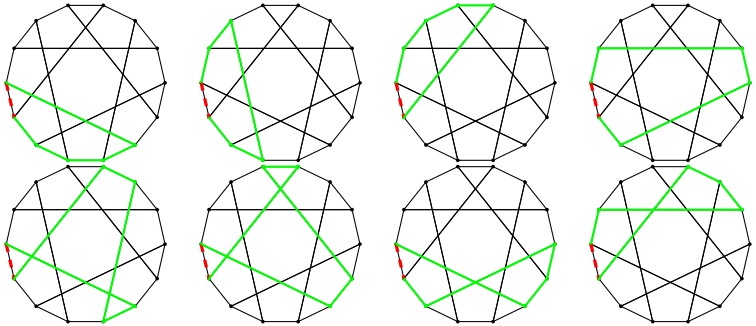


Figure: The $(3, 6)$-cage: each edge is contained in 8 girth cycles and thus each vertex is contained in 12 girth cycles.

1 Vertex-girth-regular graphs

Def.: a *vertex-girth-regular* (v, k, g, λ) graph (abbreviated as $\mathrm{vgr}(v, k, g, \lambda)$) is a (v, k, g) graph in which every vertex is contained in λ girth cycles.



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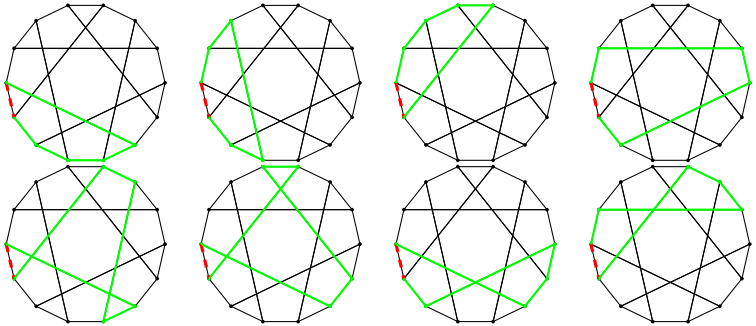
1 Vertex-girth-regular graphs

Goal: this motivates us to study $n(k, g, \lambda)$, which is the smallest order v such that a $\mathrm{vgr}(v, k, g, \lambda)$ graph exists (e.g. $n(3, 6, 12) = 14$).



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- Vertex-girth-regular graphs: the sum of entries in $\mathbf{a}(u)$ equals the sum of entries in $\mathbf{a}(v)$ for all vertices u, v
- **Vertex-girth-regular graphs are the most general of all these**

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Proposition (Jajcay, J., Porupsánszki)

Let G be a $\mathrm{vgr}(v, k, g, \lambda)$-graph. Then $\lambda \leq nc(k, g) := \frac{k(k-1)^{\lfloor \frac{g}{2} \rfloor}}{2}$ with equality if and only if G is a Moore graph.

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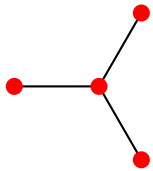

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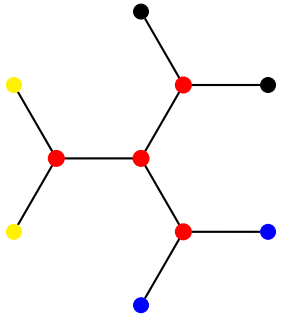

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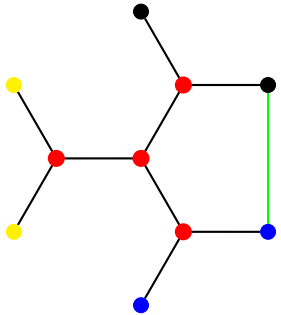

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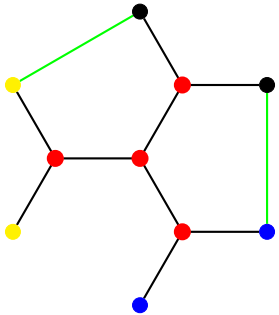

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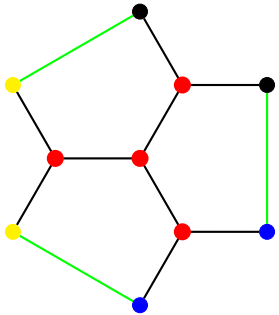

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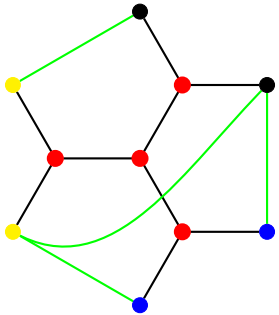

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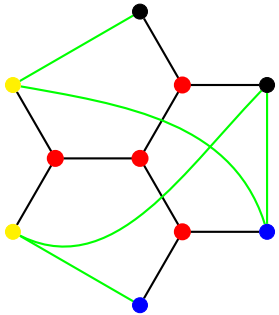

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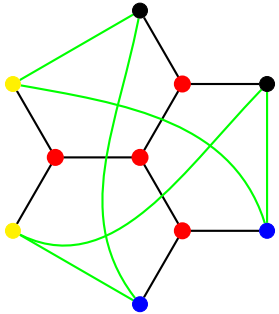

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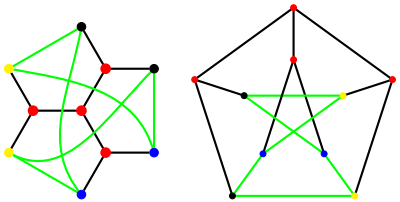
This graph is in fact isomorphic with the Petersen graph ($k = 3$, $g = 5$, $\lambda = 6$).



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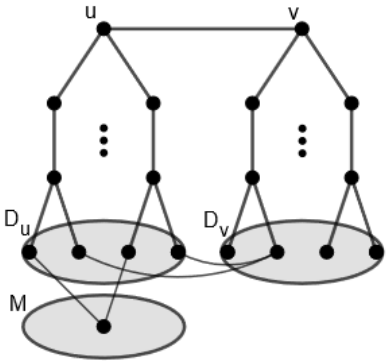

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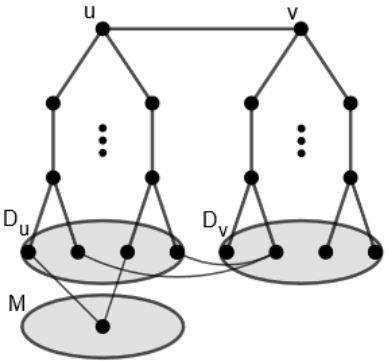

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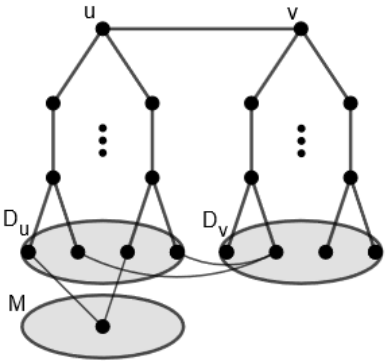

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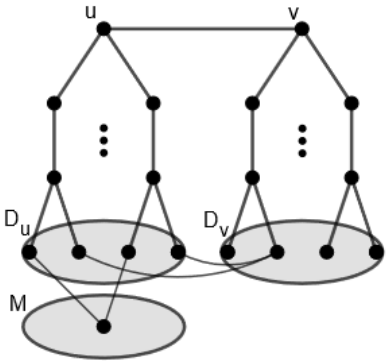

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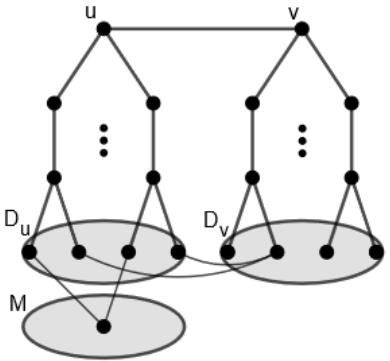

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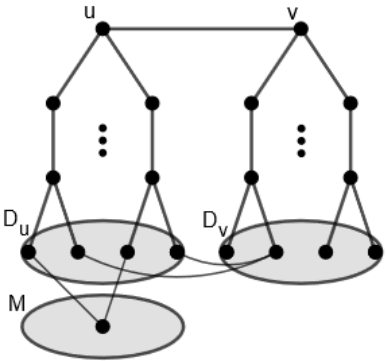

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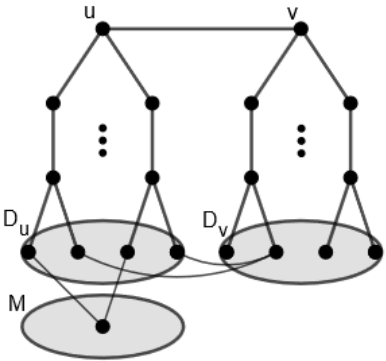

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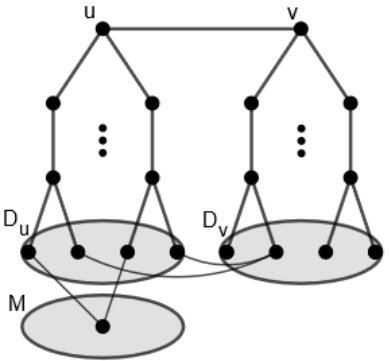

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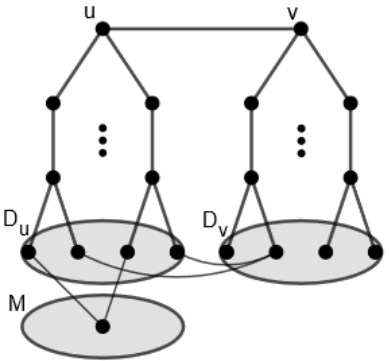

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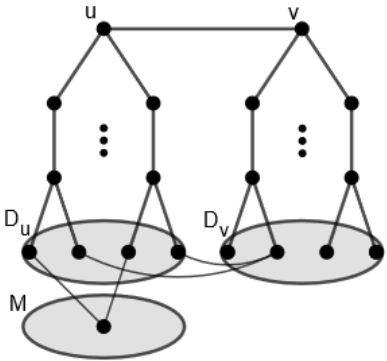

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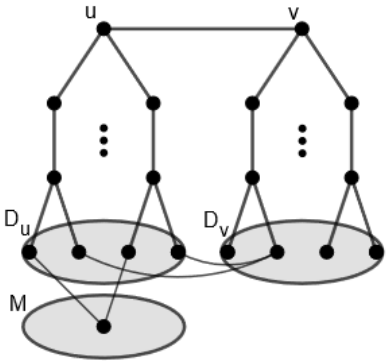

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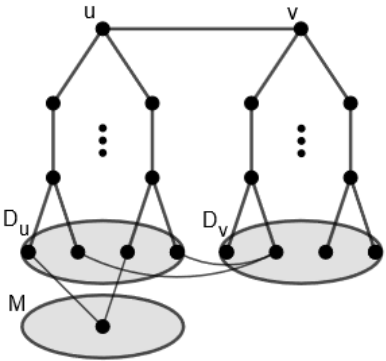

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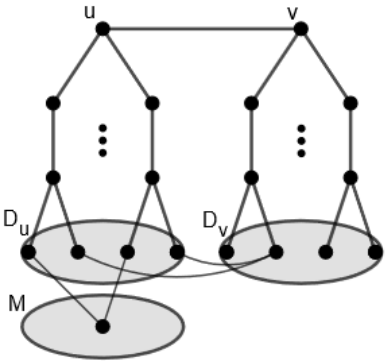
- Summing everything yields the upper bound: $\lambda \leq nc(k,g)$
- Equality can only occur when $|\mathcal{A}_u|$ and $|\mathcal{B}_u|$ are maximal, in which case the graph must be a Moore graph



2 Non-existence

- A similar set-up can be used to show that a $vgr(v, k, g, \lambda)$ graph does not exist when λ is too close (but not equal) to its maximal value (i.e., $nc(k, g) := \frac{k(k-1)^{\lfloor \frac{g}{2} \rfloor}}{2}$)

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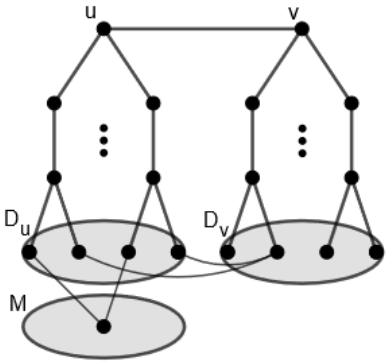
Theorem (Jajcay, J., Porupsánszki)

Let $k \geq 3$, $g = 2s \geq 4$, and $0 < \epsilon < k - 1$ be integers. Then there is no $\mathrm{vgr}(n, k, g, nc(k, g) - \epsilon)$-graph.

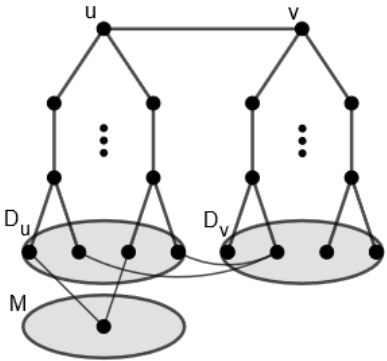
2 Non-existence proof

- Since $\epsilon > 0$, at least one horizontal edge is missing



2 Non-existence proof

- Since $\epsilon > 0$, at least one horizontal edge is missing
- How large can the number of girth cycles be now?



2 Non-existence proof

Maximize:
$\sum_{w \in D_v} y + \sum_{w \in D_v} \binom{y_w}{2} + \sum_{w \in M} \binom{x_w}{2}$
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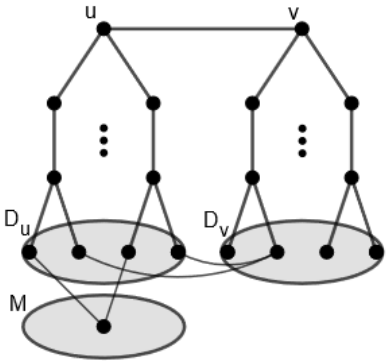

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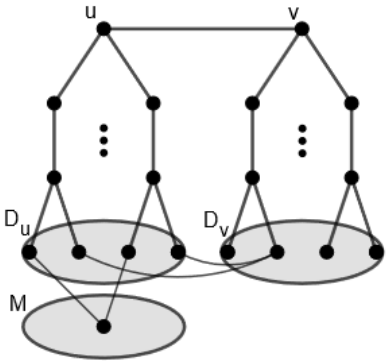

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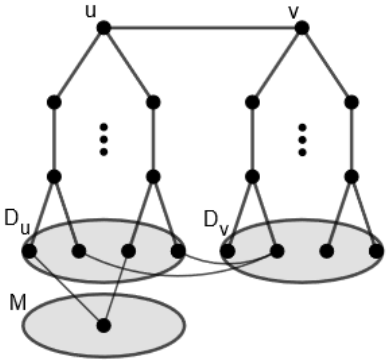

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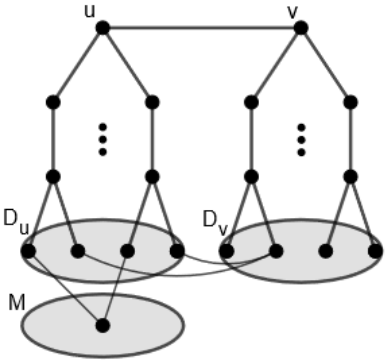

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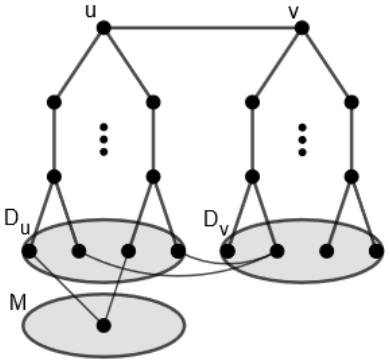

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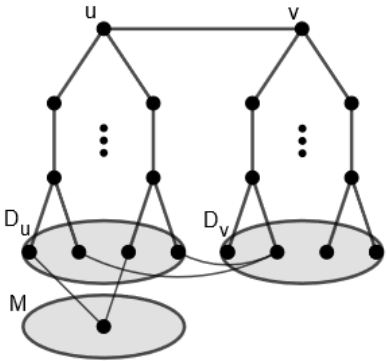
- This expression is maximized when $y_w = k - 1$ for all except for one $w \in D_v$, in which case $y_w = k - 2$.



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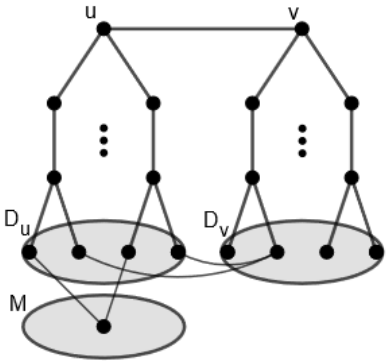
- Filling in the numbers then completes the proof



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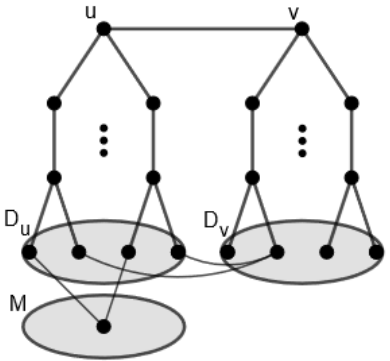
- I will only briefly comment on one of the combinatorial bounds (without going into detail) for the even girth case

2 Lower bounds on $n(k, g, \lambda)$

- In any $\mathrm{vgr}(v, k, g, \lambda)$ graph, $v \geq M(k, g) + |M|$



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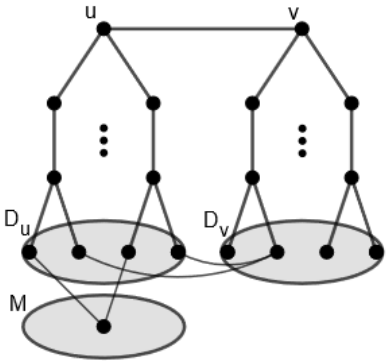

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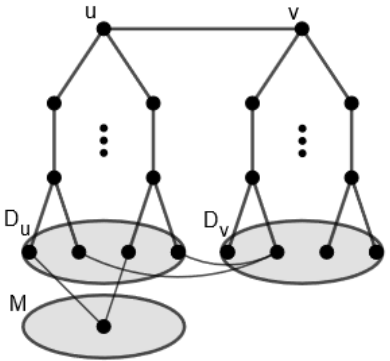

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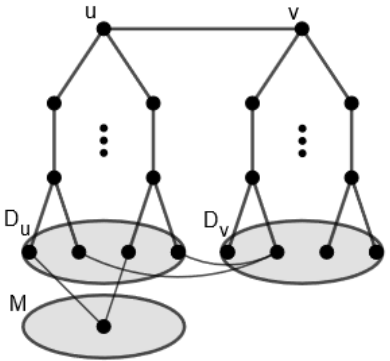
- $nc(k, g) - \lambda$ tells us something about how many horizontal edges must at least be missing
- If λ is very small, many edges must exist between D_u and M
- However, it is impossible that many of these edges go to the same vertex in M, because this would yield many girth cycles



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- Idea is based on generalized truncation: replace each vertex of degree d with a suitable graph of order d
- Same underlying idea can be used to prove:

Proposition (Jajcay, J., Porupsánszki)

If a $\mathrm{vgr}(v, k, g, \lambda)$*-graph exists, then there exist infinitely many integers* v' *such that a* $\mathrm{vgr}(v', k+1, g, \lambda)$*-graph also exists.*

2 Edge-girth-regular graphs yield vertex-girth-regular graphs

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Several families of edge-girth-regular graphs are known, as well as operations yielding new edge-girth-regular graphs

2 Existence results

Combining generalized truncation with already existing ideas from edge-girth-regular graphs, one can obtain that vertex-girth-regular graphs must exist for many (k, g, λ):

Theorem (Jajcay, J., Porupsánszki)

There are infinitely many integers v such that a $\mathrm{vgr}(v, k, g, \lambda)$-graph exists:

(i) *for $\lambda = 1$ and all integers $k, g \geq 3$;*

(ii) *for $\lambda = 2$ and all integers $k \geq 4$, $g \geq 3$;*

(iii) *for all integers $\lambda \geq 3$, $k \geq \lambda$, $g \geq 3$, $g \notin \{4, 5\}$.*

3 Outline

3 From $n(k, g, \lambda)$ to exhaustive generation

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- If the algorithm found a $\mathrm{vgr}(v, k, g, \lambda)$ graph, then clearly $n(k, g, \lambda) \leq v$.
- If the algorithm did not find any $\mathrm{vgr}(v', k, g, \lambda)$ graph for all $v' < v$, then $n(k, g, \lambda) \geq v$, because the algorithm is exhaustive.

3 A naive approach

Problem

Given integers v, k, g and λ, enumerate all $\mathrm{vgr}(v, k, g, \lambda)$ graphs.

There exist efficient generation algorithms for exhaustively enumerating all k-regular graphs with girth g of order v.

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Order	Nb. 3-regular	Nb. 3-regular and vgr
$n = 18$	$41,301$	10
$n = 20$	$510,489$	44
$n = 22$	$7,319,447$	3

3 Exhaustive enumeration algorithm

Algorithm for enumerating all $\mathrm{vgr}(v, k, g, \lambda)$ graphs:

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This basic strategy is extended with pruning rules and heuristics that are very important to obtain a fast algorithm.

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Heuristics:

- Order in which edges are added does not matter
- Add next edge adjacent to vertex with fewest options (fail-first principle)

3 Efficient algorithms for subproblems: girth cycles

Problem

For a graph G of girth g and edge $e \in E(G)$, determine the number of cycles of length g that contain e.

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Instead: linear time algorithm based on dynamic programming

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Observation. *Let u_1u_2 be an edge of the graph $G = (V, E)$ with girth g. Now $ngc(G, u_1u_2) = 0$ if $d(G - u_1u_2, u_1, u_2) > g - 1$ and $ngc(G, u_1u_2) = nsp(G - u_1u_2, u_1, u_2)$ otherwise.*

3 Efficient algorithms for subproblems: girth cycles

Observation. *Let* $D_i = \{u \in V(G) \mid d(G - u_1u_2, u, u_1) = i\}$ *be the set of vertices at distance* i *from* u_1 *in the graph* $G - u_1u_2$*. We now have the following base case (for* $u = u_1$*):*

$$nsp(G - u_1u_2, u_1, u_1) = 1$$

For a vertex $u \neq u_1$*, let* $d(G - u_1u_2, u, u_1) = i > 0$*. Since every shortest path of length* i *can be obtained by adding an edge to a shortest path of length* $i - 1$*, we have the following recursive case:*

$$nsp(G - u_1u_2, u, u_1) = \sum_{u' \in N_{G-u_1u_2}(u) \cap D_{i-1}} nsp(G - u_1u_2, u', u_1)$$

3 Efficient algorithms for subproblems: distance update

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For a graph G, where $d(G, u, v)$ is known for each pair of vertices u, v, efficiently determine $d(G + ab, u, v)$ for each pair of vertices u, v when an edge ab is added.

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Idea 2: $d(G + ab, u, v) =$
$\min(d(G, u, v), d(G, u, a) + 1 + d(G, b, v), d(G, u, b) + 1 + d(G, a, v))$.
Distance update in $O(1)$ time per pair u, v.

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Problem

For a graph G, determine if the recursive function was already called with a graph G' isomorphic with G.

Can we just compare the adjacency matrices?

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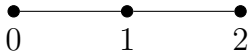

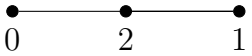

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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Use *nauty* for computing canonical form of graphs and store in appropriate data structure.
For example, array containing 10^6 canonical forms would require 10^6 comparisons in the worst case to solve the above problem.

3 Efficient algorithms for subproblems: isomorphism

Problem

For a graph G, determine if the recursive function was already called with a graph G' isomorphic with G.

Use *nauty* for computing canonical form of graphs and store in appropriate data structure.
For example, array containing 10^6 canonical forms would require 10^6 comparisons in the worst case to solve the above problem.
A standard data structure such as a splay tree only needs around 20 comparisons (in general: insertion and lookup in $O(\log(n))$ time).

3 Computational results

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- We now discuss a few interesting observations based on these calculations

3 Influence of λ

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- All our lower bounds decrease for increasing λ
- Moore graphs occur precisely when λ attains it maximal value
- Perhaps $n(k, g, \lambda) \geq n(k, g, \lambda + 1)$?

3 The inequality can fail: $n(3,8,8) = 42 < n(3,8,9) = 48$

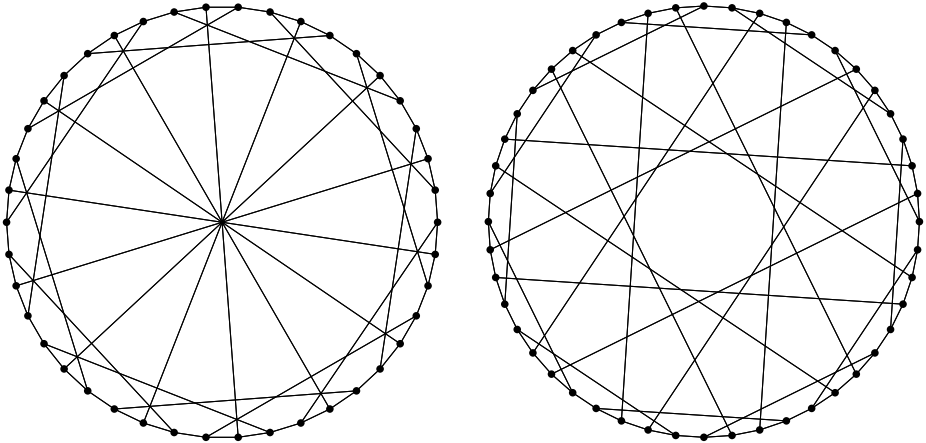

Figure: An extremal $\mathrm{vgr}(42,3,8,8)$-graph (left) and an extremal $\mathrm{vgr}(48,3,8,9)$-graph

3 Regularity versus symmetry

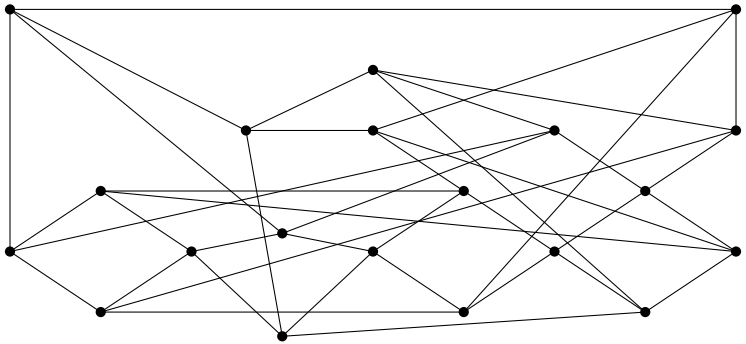

Figure: A $\mathrm{vgr}(20, 4, 4, 2)$-graph which is asymmetric (it only has the trivial automorphism).

3 Non-bipartite minimum order vgr graphs with even girth

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The analogous question for vertex-girth-regular graphs has a negative answer:

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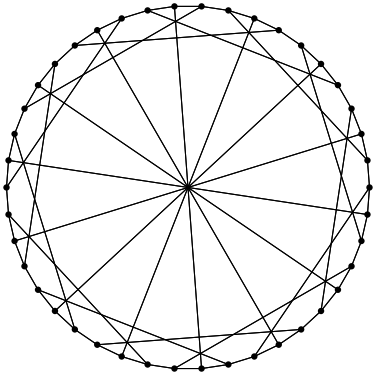

Figure: An extremal $\mathrm{vgr}(42, 3, 8, 8)$-graph

4 Outline

4 Conclusions

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- **Future work**: analyze the obtained graphs and detect patterns leading to infinite families of graphs
- **Future work**: improve upper bounds on $n(k, g, \lambda)$ based on existing constructions for cages
- **Future work**: use similar algorithimic ideas for other variants of the cage problem

Thank you!

Full paper:

Contact: jorik.jooken@kuleuven.be

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Typically, most important speed-ups come from algorithmic insights.
However, for a fixed algorithm the speed can differ an order of magnitude depending on how carefully it is implemented.

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- It might cost your sanity :-)

4 Bitwise programming

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- Hardware supports many standard operations on bitsets (set intersection, set union, maximum, minimum, ...) using only $\frac{n}{64}$ elementary operations as opposed to n.
- Many "graph theory operations" (e.g. do two vertices have a common neighbor?) can be sped up by factor of 64.

4 Shameless (self-)advertising

For people interested in knowing more about computer-aided graph theory, some relevant people to look up:

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4 Hardware infrastructure

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- Google, Amazon, ...: > 1000 euros per CPU-year (I think, never used it myself)