Resolving Sets Breaks Symmetry in Graphs

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Suppose there is a blind man and he wants to use his keys to open the doors of the house. He can find the key of each door only by touching the keys. So the labels should be such that they can be identified by touch.

What is the minimum number of labels with which we can distinguish the keys for a blind man?

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Two decades later, when Albertson and Collins, studied and applied this concept, it was added to the graph theory literature with the names of *distinguishing labeling* or *asymmetric coloring*.

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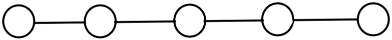
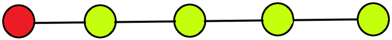
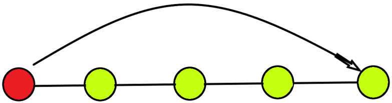
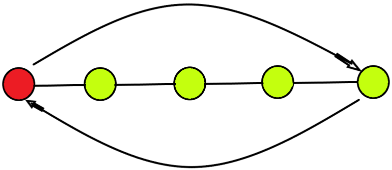
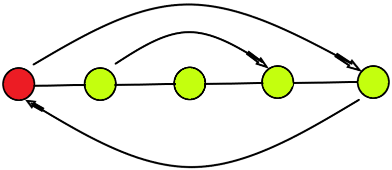
We denote the minimum such ℓ by $D(G)$ and is called distinguish number of G.

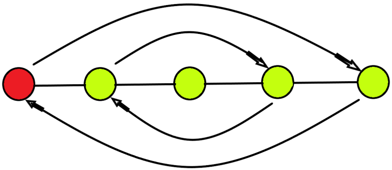


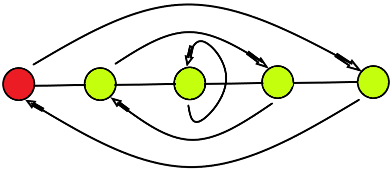






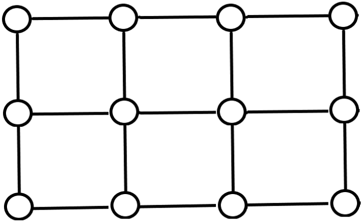






- $D(P_n) = 2$

- $D(P_3 \square P_4) = ?$



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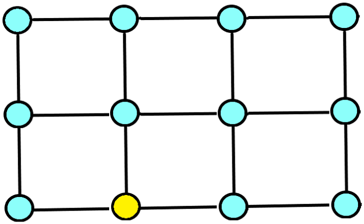

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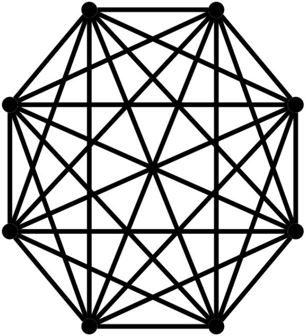
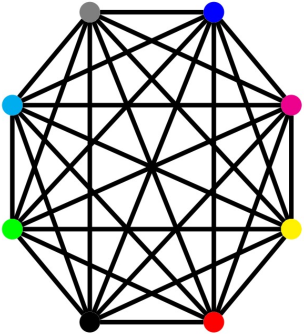


- $D(G) = n \iff G \in \{K_n, \overline{K_n}\}$.



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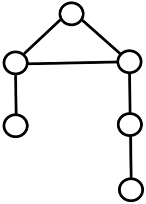
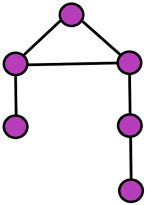
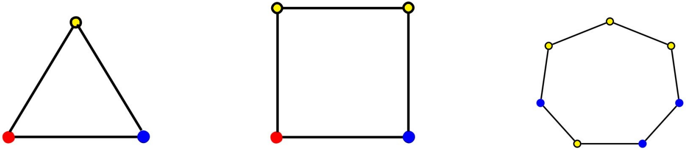


- $D(G) = 1 \iff G$ is asymmetric, i.e, $\mathrm{Aut(G)} \cong \{\mathrm{id}\}$.



- $D(C_n) = \begin{cases} 3, & \text{if } n \leq 5 \\ 2, & \text{if } n \geq 6 \end{cases}$



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- (Collins and Trenk 2006):
  - For any tree T, $D(T) \leq \Delta(T)$.

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- For total colorings: Kalinowski, Pilsniak and Woniak (2016)

Metric Dimension

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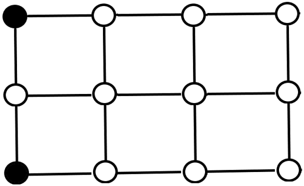
For an ordered subset $S = \{v_1, v_2, \ldots, v_k\}$ of vertices and a vertex v in a connected graph G, the metric S-representation of v is the vector $r(v|S) = (\mathrm{d}(v, v_1), \ldots, \mathrm{d}(v, v_k))$. The set S is a resolving set for G if every two vertices of G have distinct S-representations. Particularly, a vertex $x \in V(G)$ resolves a pair of vertices $v, u \in V(G)$ if $\mathrm{d}(v, x) \neq \mathrm{d}(u, x)$. The minimum cardinality of a resolving set of G is called metric dimension of G and denoted by $\dim(G)$.

- $\dim(G) = 2$



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D. Kuziak, I.G. Yero, Metric dimension related parameters in graphs: A survey on combinatorial, computational and applied results, (2021). arXiv:2107.04877 [math.CO]

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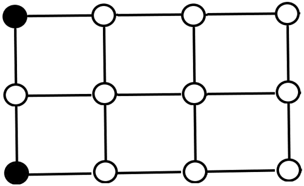
Corollary

For any connected graph G of order n, $D(G) \leq n - \operatorname{diam}(G) + 1$.

Outline of proof



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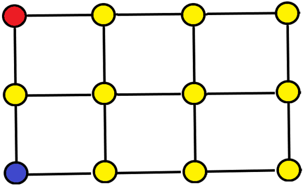

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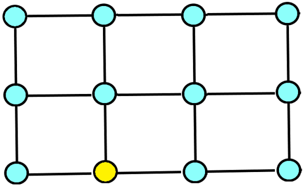

Theorem

For every $1 \leq n < m$, there exists a graph G having distinguishing number n and metric dimension m.

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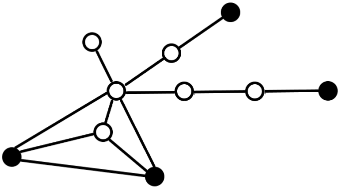

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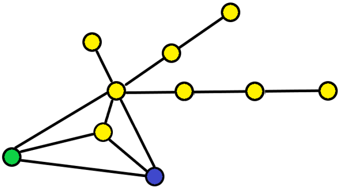

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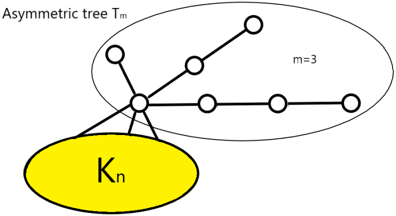

Problem

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Characterize the class A of graphs such that $G \in A$ if and only if $D(G) = \dim(G) + 1$.

M. Korivand and N. Soltankhah. A Connection between Metric Dimension and Distinguishing Number of Graphs. arXiv:submit/5294657 [math.CO] 14 Dec 2023

Call me!

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Thanks

'Thanks for your attention'