Schur Rings as a Tool for Algebraic Graph Theory: An example involving GRRs

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Introduction

- Mostly expository to illustrate that even a straightforward use of Schur rings can help answer graph theoretic questions.
- Basic introduction to Schur rings so maybe a bit boring for those already familiar.
- Basic Introduction to Graphical Regular Representations.

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- Coherent configurations (late 1960s): D.G. Higman, mostly to tackle problems in group theory and simultaneously Weisfeiler and Leman, mainly to tackle the graph isomorphism problem.

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The group ring $\mathbb{Z}[G]$

Let G be a finite group. Then the group ring $\mathbb{Z}[G]$ consists of all formal linear combination

$$\sum_{g \in G} \lambda_g \cdot g.$$

Addition and multiplication in $\mathbb{Z}[G]$ is carried out in the natural way.

Example

Let $G = \langle a, b : a^2 = b^7 = 1, ba = a^{-1}b \rangle$.

Let $r = 2a + b$, and $s = a + 3b$.

Then

$r + s = 3a + 4b$.

And

$rs = (2).1 + 6ab + ab^6 + 3b^2$

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Notation

If $T = \{t_1, t_2, \ldots, t_p\} \subseteq G$, then the element

$$t_1 + t_2 + \ldots + t_p$$

of $\mathbb{Z}(G)$ is denoted by

$$\underline{T}.$$

The element $\underline{T}$ is said to be a *simple quantity*

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Definition of a Schur ring

A subring $\mathcal{S}$ of the group ring $\mathbb{Z}[G]$ is called a *Schur ring* $\mathfrak{S}$ or an $\mathcal{S}$-ring over G, of rank r if the following conditions hold:

- $\mathcal{S}$ is closed under addition and multiplication incuding multiplication by elements of $\mathbb{Z}$ from the left (i.e. $\mathcal{S}$ is a $\mathbb{Z}G$-module);
- Simple quantities $\underline{T}_0, \underline{T}_1, \ldots, \underline{T}_{r-1}$ exist in $\mathcal{S}$ such that every element $\sigma \in \mathcal{S}$ has a unique representation;

$$\sigma = \sum_{i=0}^{r-1} \sigma_i \underline{T}_i$$

- $\underline{T}_0 = \underline{1}$, $\sum_{i=0}^{r-1} \underline{T}_i = \underline{G}$, that is, $\{T_0, T_1, \ldots, T_{r-1}\}$ is a partition of G;
- For every $i \in \{0, 1, 2, \ldots, r-1\}$ there exists a $j \in \{0, 1, 2, \ldots, r-1\}$ such that $\underline{T}_j = \underline{T}_i^{-1} (= \{x^{-1} : x \in T_i\})$.

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Basic elements

We call the simple quantities $\underline{T}_0, \underline{T}_1, \ldots, \underline{T}_{r-1}$ the *basis* of the Schur ring; the $\underline{T}_i$ its *basic elements* and the T_i its *basic sets*.

Example

Let Γ be a group of automorphisms of the group G. Then the orbits of G under the action of Γ form the basic sets of a Schur ring over G.

In particular, the conjugacy classes of G form the basic sets of a Schur ring over G.
(The basic elements would here be the well-known "class sums" so important in linear representations of groups.)

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Structure constants

Because of closure under multiplication, the product of two linear combinations of $\underline{T}_0, \underline{T}_1, ..., \underline{T}_{r-1}$ must also be a linear combination of these simple quantities.
Therefore, for $i, j \in \{1, ..., r\}$, there exist non-negative integers p_{ij}^k called structure constants, such that

$$\underline{T}_i \cdot \underline{T}_j = \sum_{k=1}^{r} p_{ij}^k \underline{T}_k$$

We shall soon see that the structure constants have a very nice graph theoretic interpretation.

Example

Let G be the cyclic group $\langle a : a^8 = 1 \rangle$. The following simple quantities are the basic sets of a Schur ring of G:

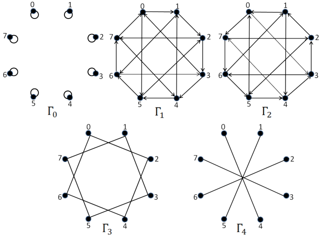
$$\{\underline{1}, \underline{a^1, a^5}, \underline{a^3, a^7}, \underline{a^2, a^6}, \underline{a^4}\}.$$

One can verify that, if we let
$T_0 = \{1\}$, $T_1 = \{a^1, a^5\}$, $T_2 = \{a^3, a^7\}$, $T_3 = \{a^2, a^6\}$, and
$T_4 = \{a^4\}$, then, for example, $p_{24}^3 = 0$.
This means that, when the product $\underline{T_2} \cdot \underline{T_4}$ is written out as a linear combination of the basic sets, the coefficient of $\underline{T_3}$ in this linear combination is 0.

Graph theoretic interpretation

Given a Schur ring we can construct a Cayley (di)graph Cay(G, T_i) with each of the basic sets T_i of the Schur ring.
For the previous example, these Cayley graphs (which we call the basic Cayley graphs associated with the Schur ring), are as shown in the following figure.

GT interpretation (cont)



We can now give an interpretation to the structure constants p^i_{jk}: pick **any** edge in $\mathrm{Cay}(G, T_i)$ and count in how many you can go from one end of this edge to the other by first passing through an edge in $\mathrm{Cay}(G, T_j)$ followed by an edge in $\mathrm{Cay}(G, T_k)$.

Verify here that $p^1_{2,3} = 2$ and $p^4_{2,1} = 2$.

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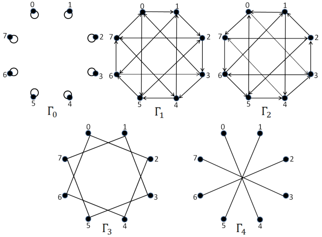

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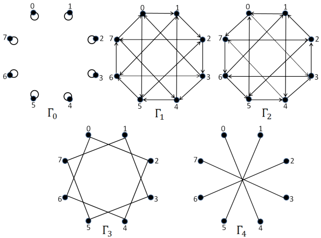

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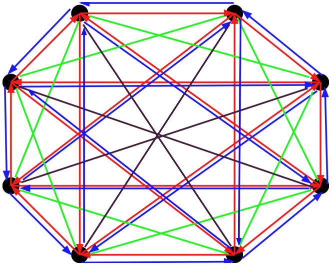
Colour graph



This colour graph (less the loops) associated with the previous Schur-ring.
If there are only two such colours and all edges are undirected, we get complementary pair of SRGs.

Verify here that $p^{red}_{blue,green} = 2 = p^{black}_{blue,red}$.

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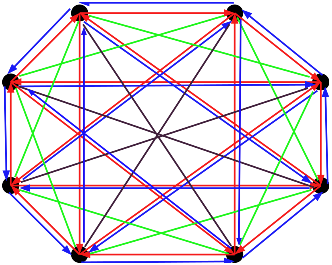

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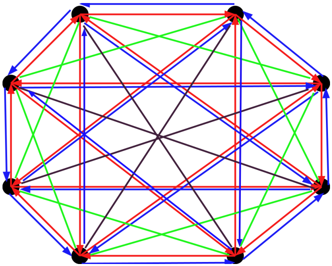

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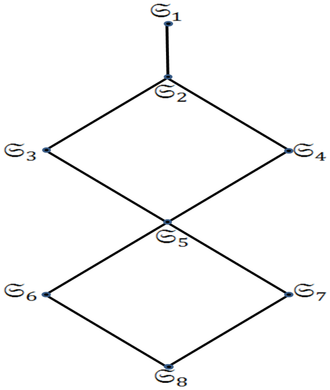
Coherent configurations

The adjacency matrices of the basic Cayley (di)graphs form what is called a coherent configuration.

The lattice of all *undirected* Schur rings for C_{27}, the cyclic group of order 27



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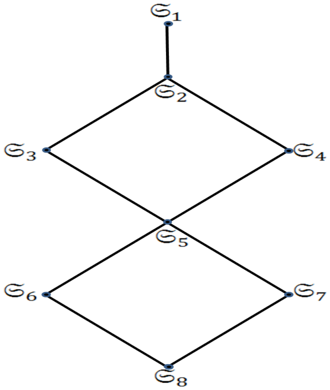

$\mathfrak{S}_1$, the "largest" Schur ring, is the finest as a partition of C_{27}, that is, the basic sets are all the singleton elements of C_{27}.
$\mathfrak{S}_8$, the "smallest", is the coarsest partition, with basic sets $\{1\}$ and $C_{27} - 1$

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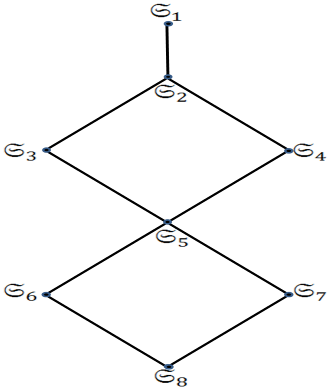

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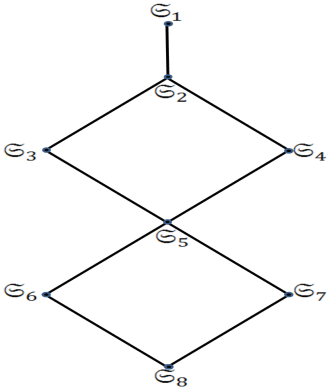

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Automorphism groups

The automorphism group of a Schur ring is defined to be the intersection of all automorphism groups of the basic Cayley graphs of the Schur ring.

The smallest (coarsest) Schur ring has the largest automorphism group, that is, the symmetric group S_n, where n is the order of the group.

The largest (finest) Schur ring has the smallest automorphism group, that is, the regular action of the group on itself.

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Main result (for our purposes)

Definition

Let S be a subset of the group G. Then $\langle\langle S \rangle\rangle$ is the smallest (coarsest) Schur ring of G containing $\underline{S}$.

Proposition

Let $\Gamma = Cay(G, S)$ be a Cayley graph of the group G. Then

$$Aut(\Gamma) = Aut(\langle\langle S \rangle\rangle)$$

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Graphical Regular Representations: GRRs

Definition

Let G be a group. Then a graph Γ is said to be a GRR of G if its automorphism group is isomorphic to G and acts regularly on $V(\Gamma)$.

Main Theorem on groups having a GRR

Theorem

1. *The only abelian groups which have a GRR are* $\mathbb{Z}_2^n$ *for* $n \geq 5$.
2. *Except for generalised dicyclic groups and a finite number of known groups, all non-abelian groups have a GRR.*

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Two important facts about GRRs

1. If Γ is a GRR of a group G then $\Gamma = \text{Cay}(G, S)$ for some connecting set S.
2. If $\Gamma = \text{Cay}(G, S)$ is a GRR of G then there is no non-trivial group automorphism ϕ of G such that $\phi(S) = S$.

Property 2 is necessary but not sufficient.

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Two important facts about GRRs

1. If Γ is a GRR of a group G then $\Gamma = \text{Cay}(G, S)$ for some connecting set S.
2. If $\Gamma = \text{Cay}(G, S)$ is a GRR of G then there is no non-trivial group automorphism ϕ of G such that $\phi(S) = S$.

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Neighbourhoods in Cayley graphs and the group ring $\mathbb{Z}[G]$

Example: Let $G = \langle a, b : a^6 = b^2 = 1, ba = a^{-1}b \rangle$ and $S = \{a^3, b, ab\}$. The the neighbourhood of vertex 1 is S.

To find the vertices which can be reached in two steps from 1 we compute $\underline{S}^2$ which comes to

$$3(1) + a + a^5 + 2a^3b + 2a^5b.$$

This means that, starting from 1, there are three ways of getting back to 1 in two steps, one way each of getting to a or a^5 in two steps, and two ways each of getting to a^3b or a^5b in two steps.

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Combining results

1. Given a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, find $\langle\langle S\rangle\rangle$, the smallest (coarsest) Schur ring of G containing $\underline{S}$.
2. Then $\mathrm{Aut}(G) = \mathrm{Aut}(\langle\langle S\rangle\rangle)$.
3. If $\mathrm{Aut}(\langle\langle S\rangle\rangle) = G$, in particular, if $\langle\langle S\rangle\rangle$ has all the singleton element subsets as basic sets, then $\mathrm{Aut}(\Gamma) = G$ and Γ is a GRR of G.

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A very useful result: Schur-Wielandt Principle

Let $r = \sum \lambda_g g$ be an element of a Schur ring $\mathcal{S}$ of a group G. Then, for any integer μ, the sum $\sum_{\lambda_g = \mu} g$ is also in $\mathcal{S}$.

Example: If $a + 2b + c + 3d + 2f$ is in $\mathcal{S}$, then so are $a + c$, $b + f$, and d.

An example

Let G be the dihedral group

$$\langle a, b : a^2 = b^7 = abab = 1 \rangle.$$

Show that $\mathrm{Cay}(G, S)$ where

$$S = \{a, ab, ab^3, b, b^6\}$$

is a GRR of G.

Example (cont)

```
gap> d7:=DihedralGroup(14);;
gap> Zd7:=GroupRing(Integers, d7);;
gap> a:=Zd7.1;
(1)*f1
gap> b:=Zd7.2;
(1)*f2

gap> a^2;
(1)*<identity> of ...
gap> b^7;
(1)*<identity> of ...

gap> (a+a*b+a*b^3+b+b^6)^2;
(5)*<identity> of ...+
(4)*f1*f2^2+
(2)*f1+(2)*f1*f2+(2)*f2^2+(2)*f1*f2^4+(2)*f2^5+(2)*f1*f2^6+
(1)*f2^3++(1)*f2+(1)*f2^4+(1)*f2^6+
```

Example (cont)

Therefore, $x = ab^2$ and $y = a + ab + b^2 + ab^4 + b^5 + ab^6$ are in $(S) = \langle\langle S \rangle\rangle$. So we multiply xy,

```
gap> a*b^2*(a+a*b+b^2+a*b^4+b^5+a*b^6);
(1)*f1+(1)*f2^2+(1)*f2^4+(1)*f1*f2^4+(1)*f2^5+(1)*f2^6
```

Therefore, $xy = a + b^2 + b^4 + ab^4 + b^5 + b^6$ is in $\mathcal{S}$

But $(xy)^{-1} = a + b^5 + b^3 + ab^4 + b^2 + b$.
But basic elements are disjoint, therefore either b or $b + b^3$ is a basic element. But by squaring the latter, we see that b^4 is a basic element, hence so is b. Multiplying x by b repeatedly it follows that so is a. Therefore $\mathcal{S}$ is the finest Schur ring on G with all singleton sets as basic sets.

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Therefore $\mathrm{Aut}(\mathrm{Cay}(G, S)) = \mathrm{Aut}(\mathcal{S}) = G$.

Our main result

Theorem

If $p > 5$ and $3r - 2s = t \mod p$ then

$$Cay(D_p, \{ab^r, ab^s, ab^t\})$$

is a GRR of D_p

Example

Let $p = 7$. Let $r = 2$, $s = 3$ and $t = 0$, therefore $3r - 2s = t$. The figure shows $Cay(D_7, \{ab^2, ab^3, a\})$.

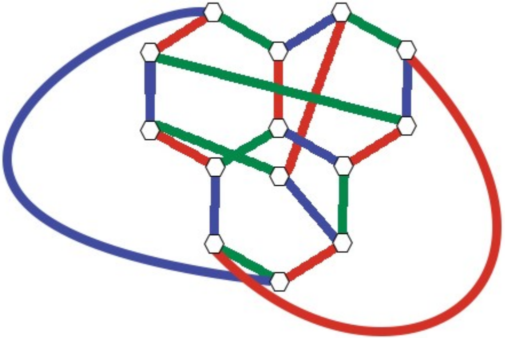

Figure: $Cay(D_7, \{ab^2, ab^3, ab\})$, a GRR of D_7.

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Second example

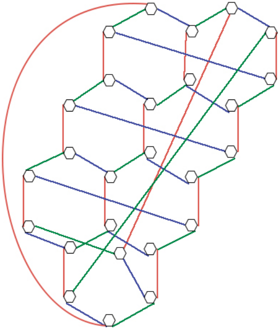

Figure: $Cay(D_{13}, \{ab, ab^3, ab^4\})$, a GRR of D_{13}.

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Work in progress

Theorem (Hopefully—still checking!)

Let m be an odd integer greater than 5 and let r, s, and t be integers less than m such that the difference of any two of them is relatively prime to m. If $3r - 2s = t \mod m$, then

$$Cay(D_{2m}, \{ab^{2r}, ab^{2s}, ab^{2t}, b^m\})$$

is a GRR of D_{2m}.

Software packages

COCO — Igor Faradzev and Mikhail Klin
Available on Andries E. Brouwer's website:
https://www.win.tue.nl/ aeb/

COCO2P — Mikhail Klin, Christian Pech*, Sven Reichard
Available at:
https://github.com/chpech/COCO2P/blob/master/README

COCO2R — Sven Reichard
https://github.com/sven-reichard/COCO2P.git

Other cubic GRRs of dihedral groups (not using Schur rings)

D. L. Powers, Exceptional trivalent Cayley graphs for dihedral groups. *J. Graph Theory*, 6 (1982), 43–55.

Other applications of Schur rings to graph theory

Enumeration of circulant graphs:
M. Klin, V. Liskovets, R. Pöschel. Analytical Enumeration of Circulant Graphs with Prime-Squared Number of Vertices. *Sèminaire Lotharingien de Combinatoire* B36d, (1996).

V. Gatt, M. Klin, J. Lauri, V. Liskovets. Constructive and analytic enumeration of circulant graphs with p^3 vertices; $p = 3, 5$. In *Isomorphism, Symmetry and Computations in Algebraic Graph Theory*. (2016) (Gareth Jones, Ilia Ponomarenko, Jozef Siran, Editors) Springer Proceedings in Mathematics & Statistics 305. (Pilsen Conference, Czech Republic, October 3–7, 2016).
https://doi.org/10.1007/978-3-030-32808-5_2
Version with full details in http://arxiv.org/abs/1512.07744

Settling Adám's Conjecture for circulant graphs:
M. Klin, R. Pöschel, The König problem, the isomorphism problem for cyclic graphs and the method of Schur rings, in: *Algebraic Methods in Graph Theory*, vol. 1, 2 (Szeged, 1978), in: Colloq. Math. Soc. János Bolyai, vol. 25, North-Holland, Amsterdam, New York, 1981, pp. 405–434.
M. Muzychuk. Ȧdám's Conjecture is True in the Square-Free Case. *J. Combin. Theory, Series A*, 72:118–134, 1995.

M. Muzychuk. On Ȧdám's Conjecture for Circulant Graphs. *Discrete Math.*, 176:285–298, 1997.

General texts and WL-stabilisation

B. Weisfeiler, *On Construction and Identification of Graphs*. Springer (1-76)

M. Muzychuk & I. Ponomarenko, Schur rings. *European Journal of Combinatorics* 30 (2009) 1526–1539.

Perhaps the ultimate use, to date, of the Weisfeiler-Leman method for graph isomorphism:

L. Babai, Graph Isomorphism in Quasipolynomial Time.
https://arxiv.org/abs/1512.03547

Thank You!