Extremal circulant graphs and where to find them

Rob Lewis, Open University, 31st March 2021

Parts of the talk

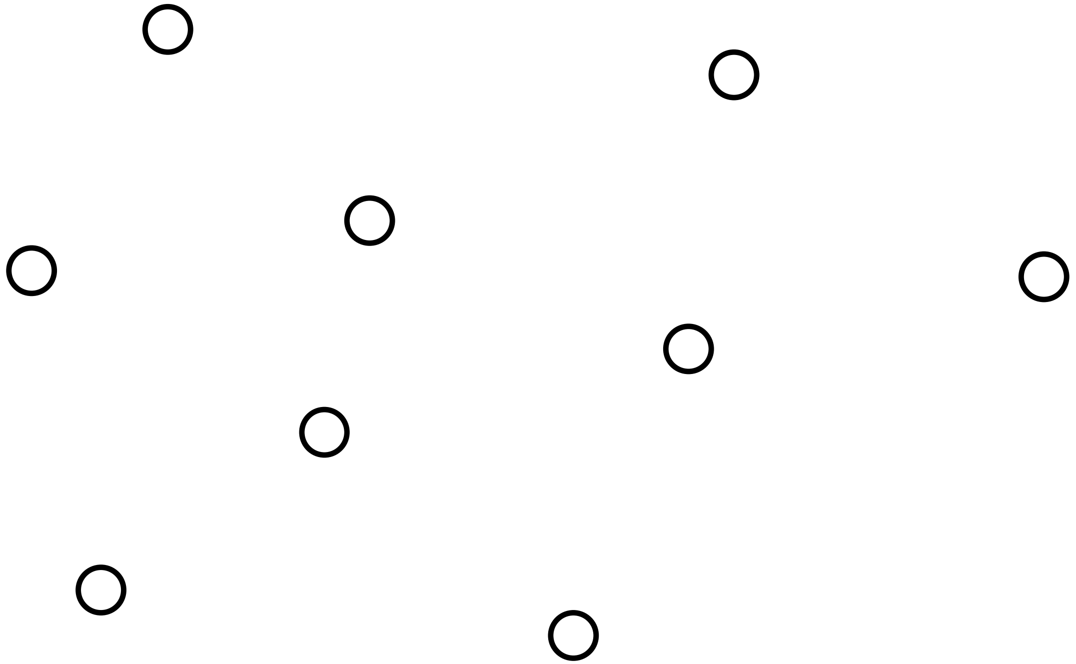
- Introduction to graphs
- The degree-diameter problem
- Extremal circulant graphs
- Where to find them

Many of the statements about circulant graphs are true for Abelian Cayley graphs in general, but for simplicity we will restrict discussion to circulant graphs.

There are a lot of slides, so if you want to ask a question about one afterwards, it would be helpful to remember the slide number

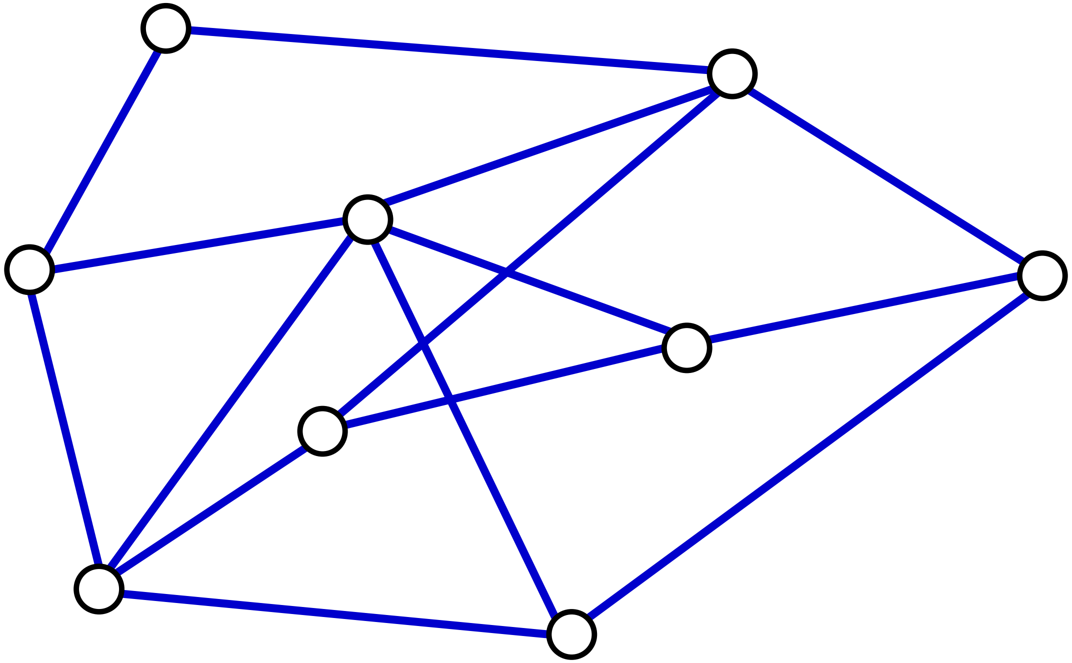
An example of a simple connected undirected graph

A graph is defined by a set of *vertices*



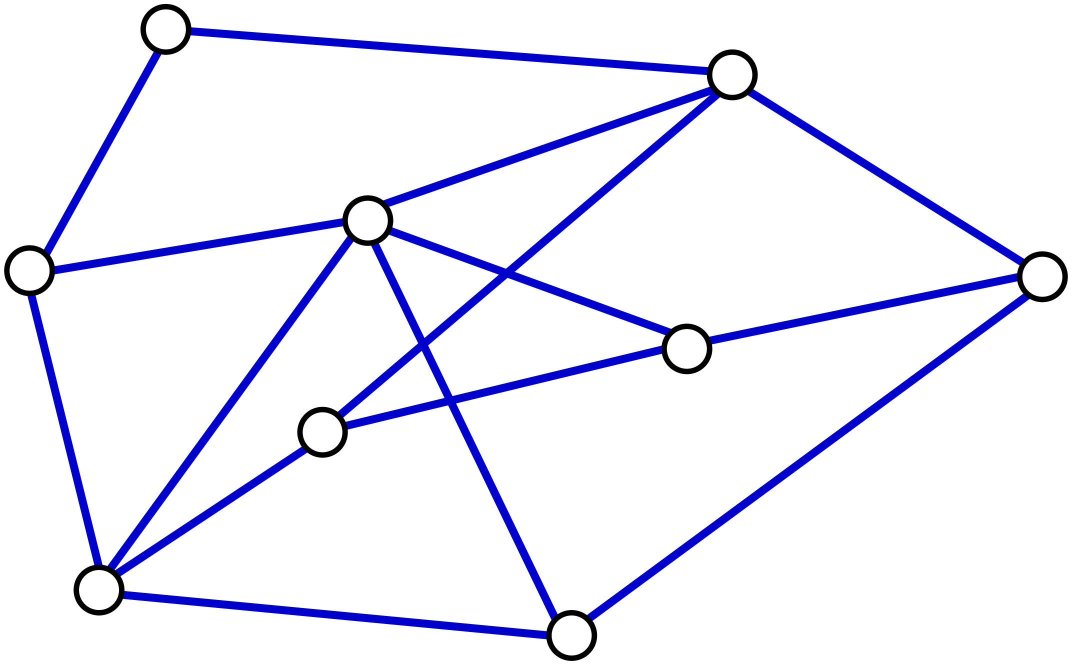
An example of a simple connected undirected graph

A graph is defined by a set of *vertices* and a set of *edges* between pairs of vertices.



An example of a simple connected undirected graph

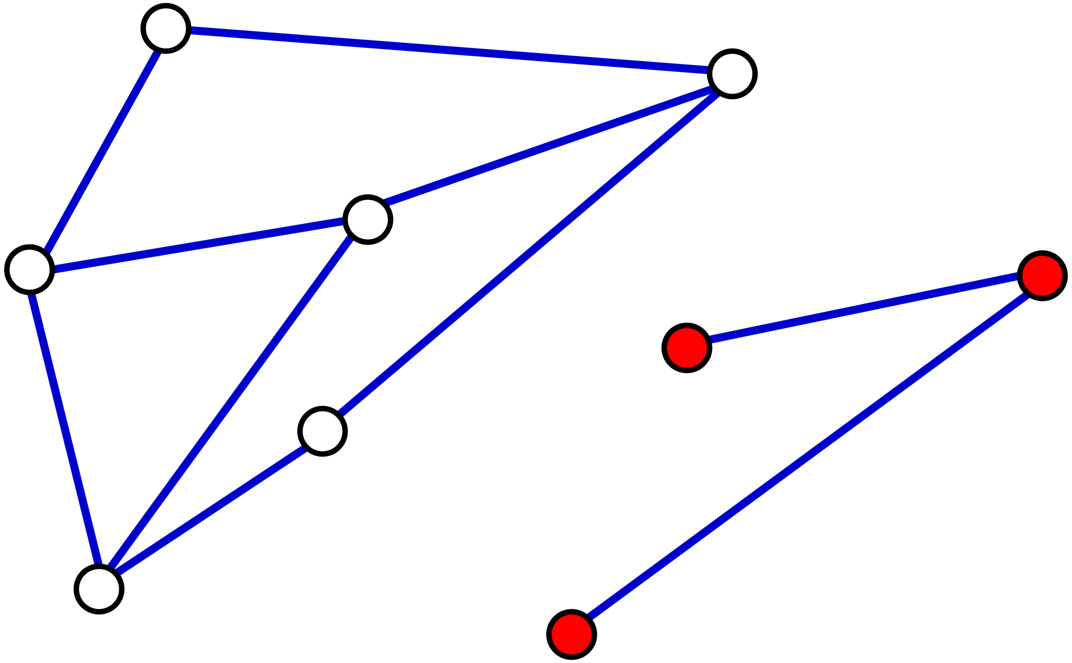
It is *connected* because there is a *path of edges* joining any two vertices.



An example of a simple connected undirected graph

It is *connected* because there is a *path of edges* joining any two vertices.

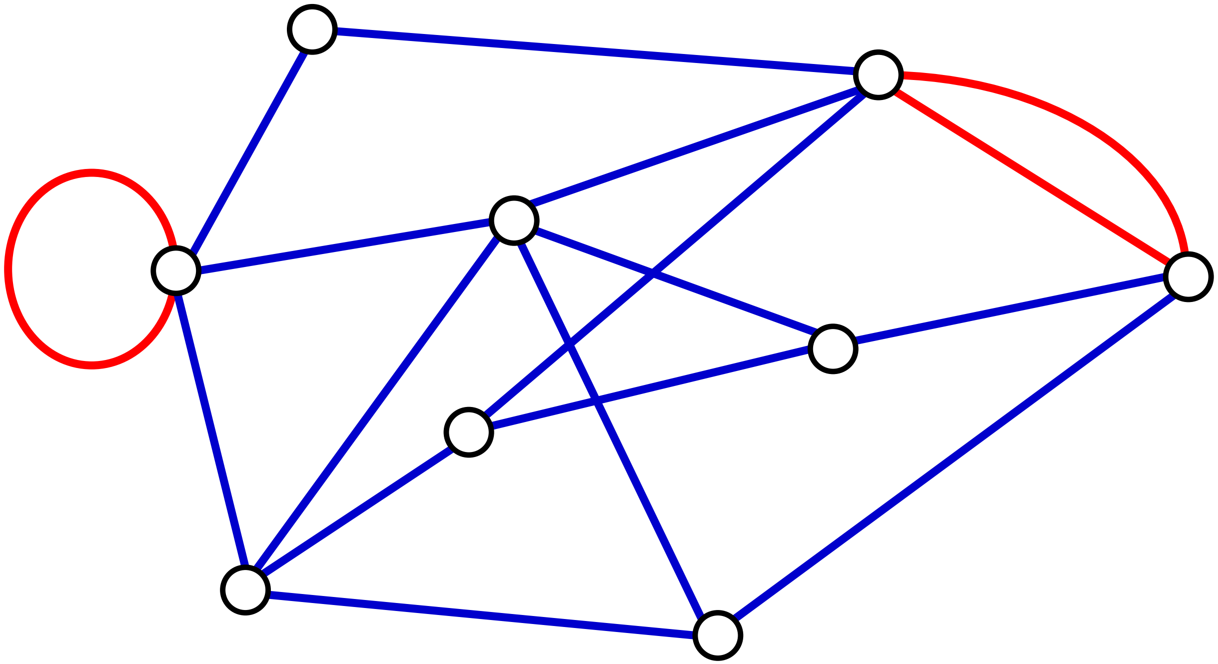
Disconnected graph with two connected components



An example of a simple connected undirected graph

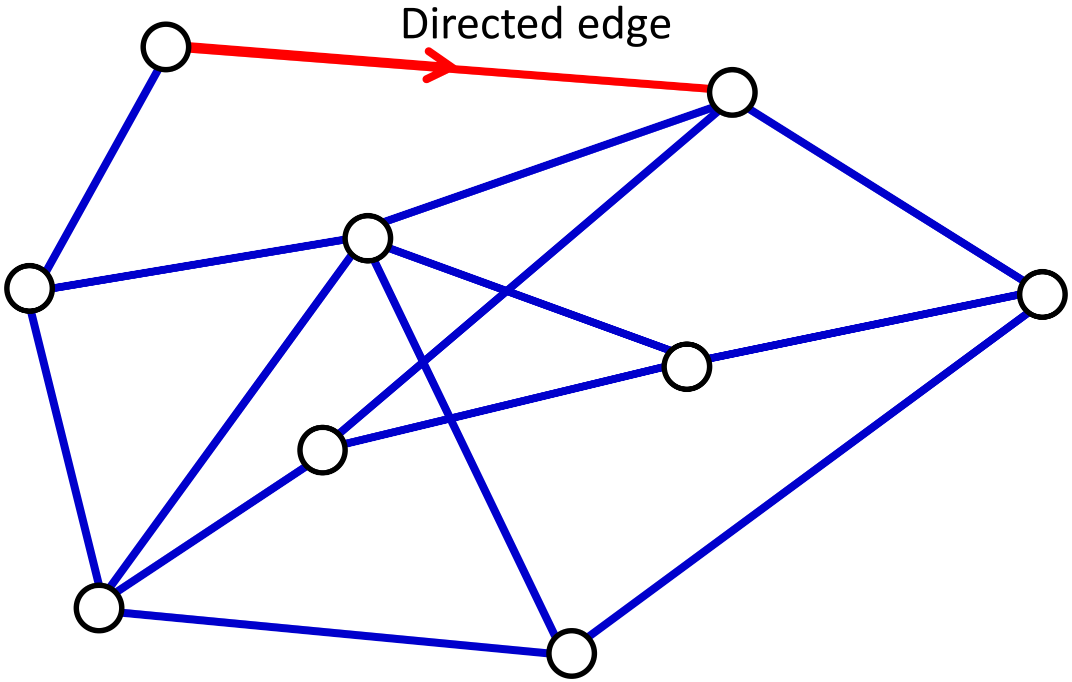
It is *simple* because there are *no loops*, where an edge connects a vertex to itself, and *no multiple edges* where two or more edges join the same pair of vertices.

Graph with a loop and a multiple edge



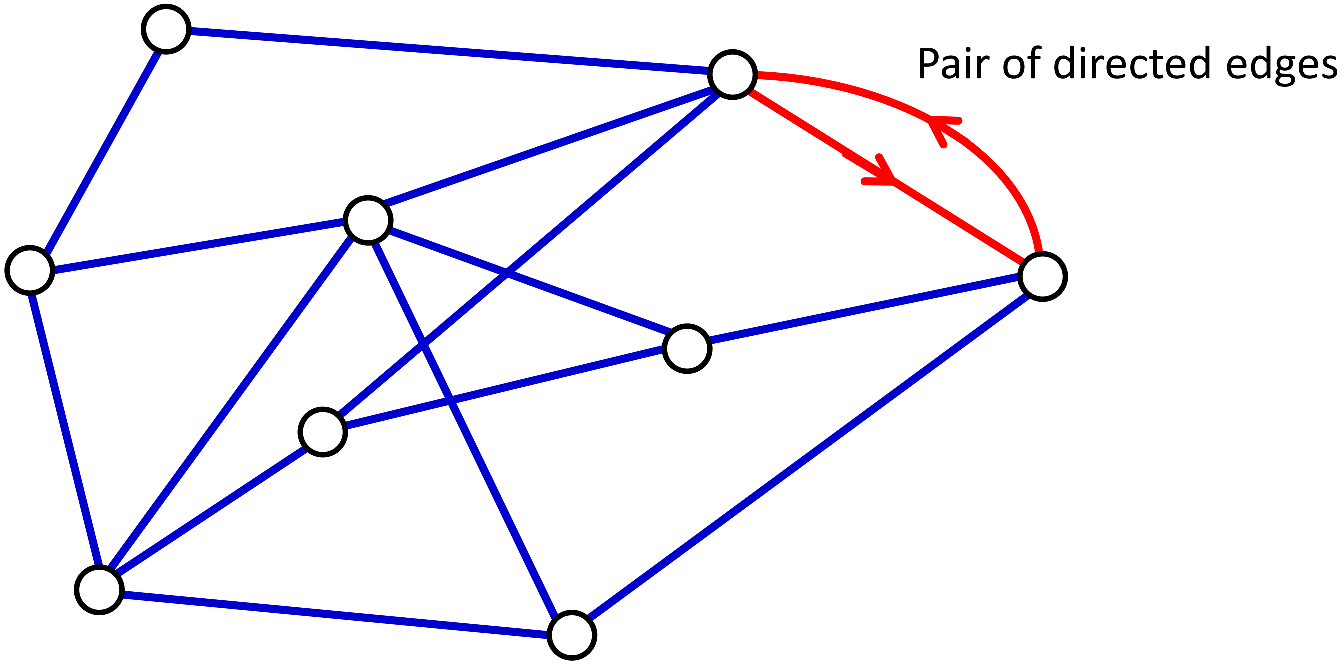
An example of a simple connected undirected graph

It is *undirected* because no edge has an assigned direction from one vertex to the other.



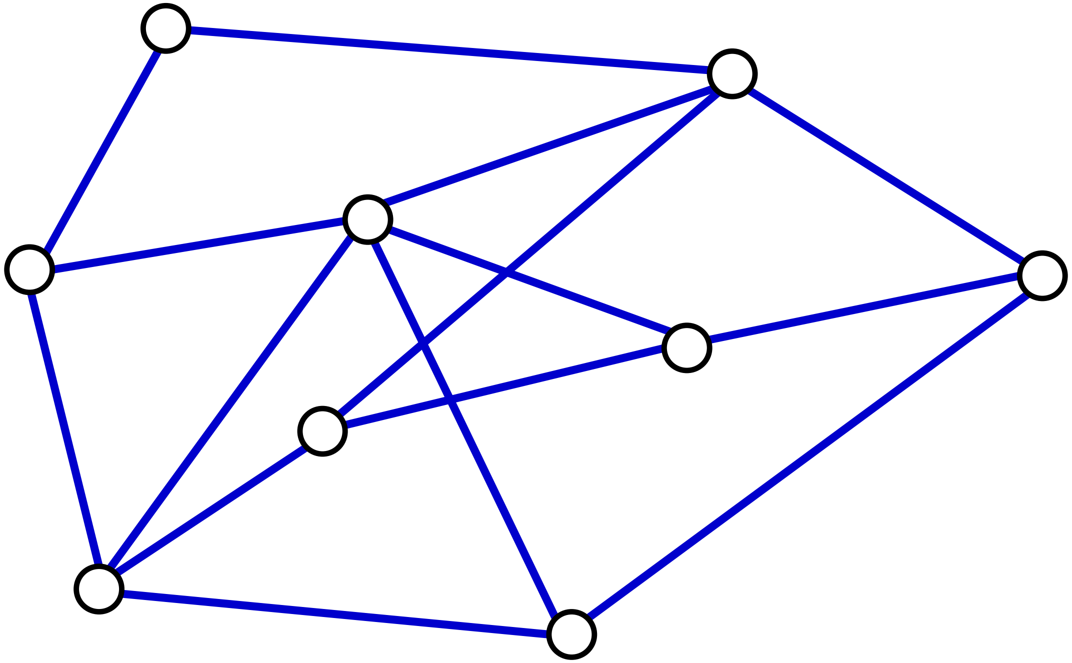
An example of a simple connected undirected graph

An *undirected* edge is equivalent to a *pair of directed edges in opposite directions*.



An example of a simple connected undirected graph

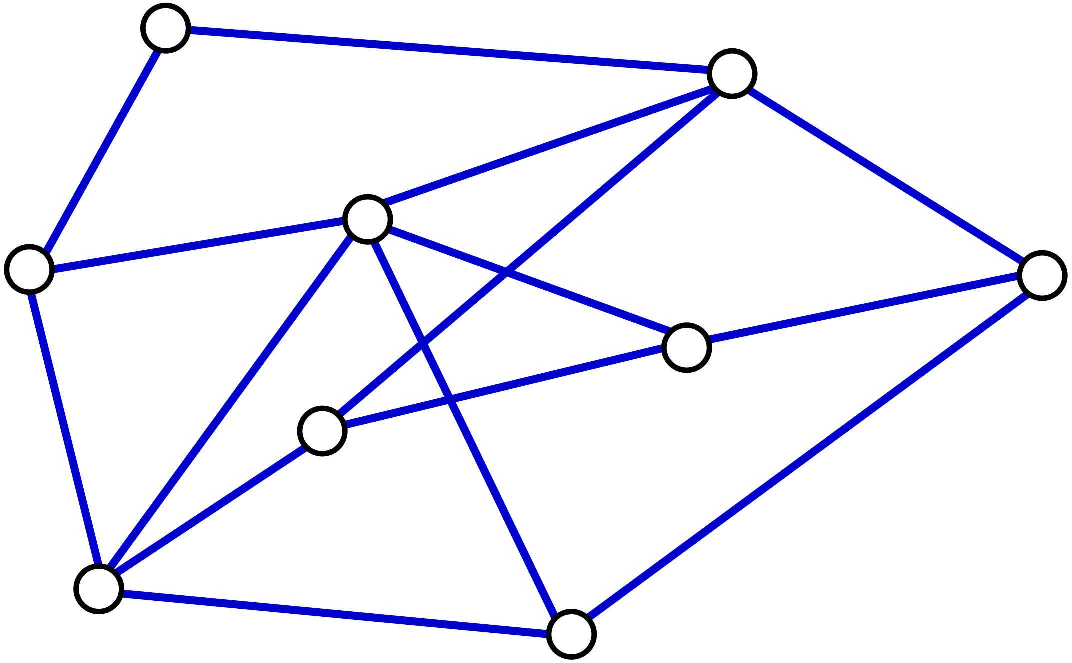
In this talk all graphs are assumed to be simple, connected and undirected.



The order of a graph

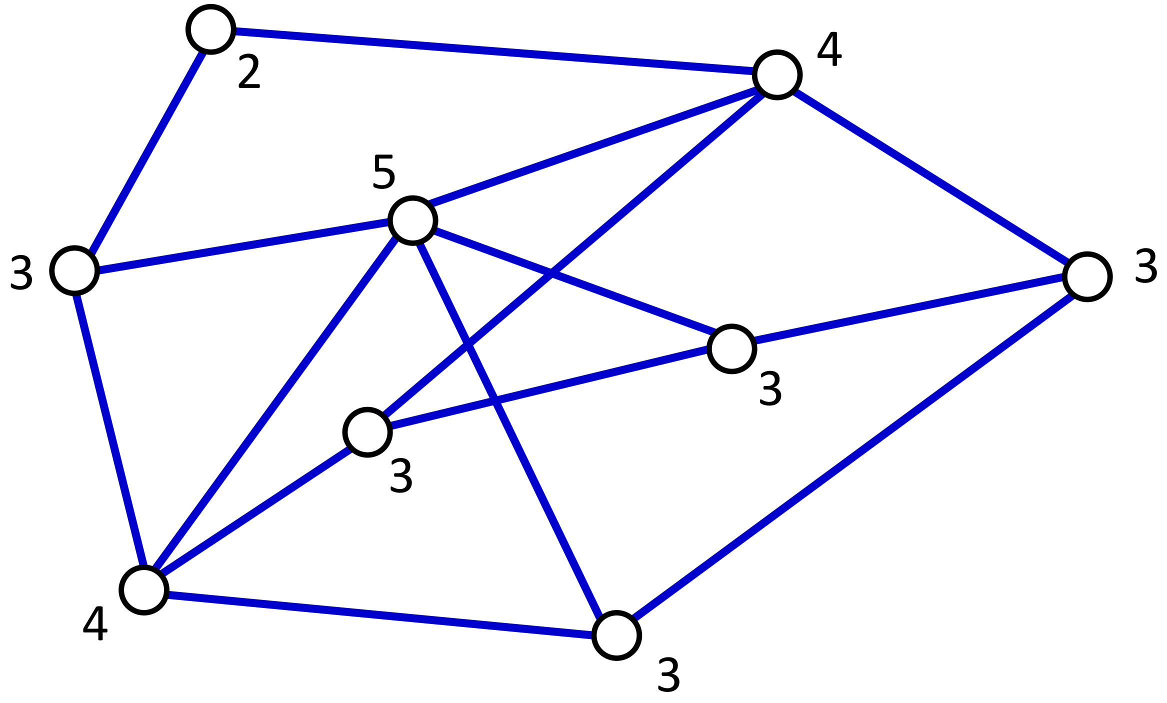
The *order* of a graph is its number of vertices.

Graph of order 9



The degree of a graph

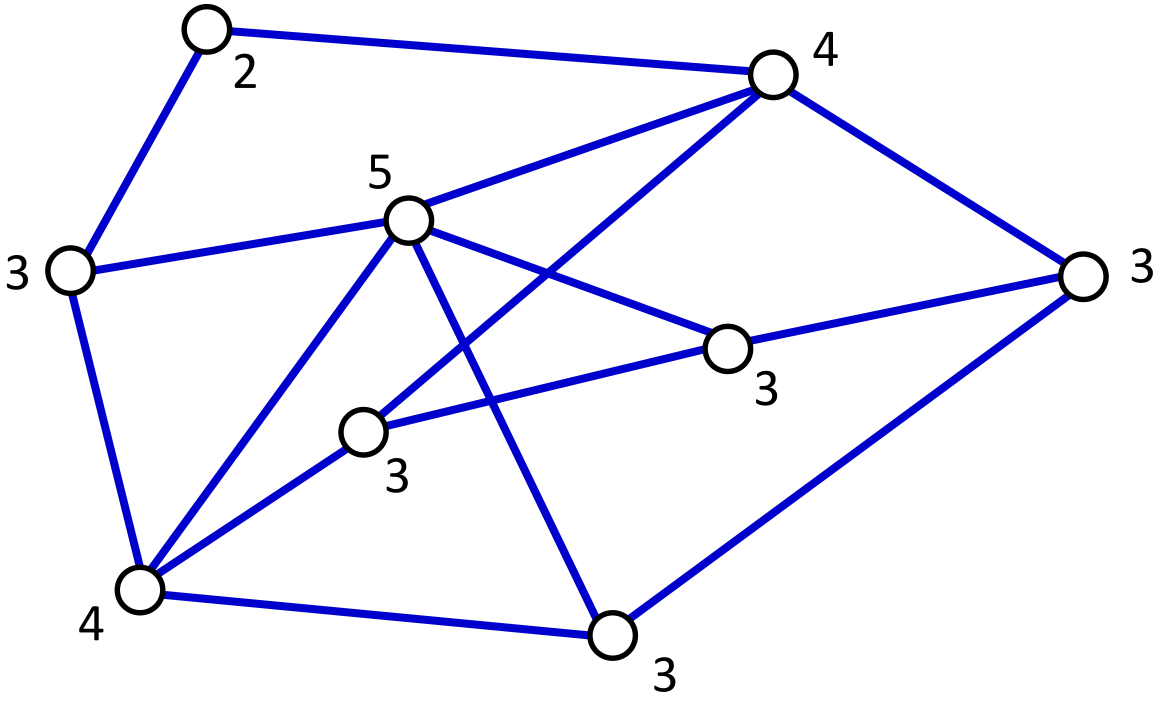
The *degree* or *valency* of a vertex is the number of edges incident to it.



The degree of a graph

The *maximum degree* of a graph is the maximum degree of its vertices.

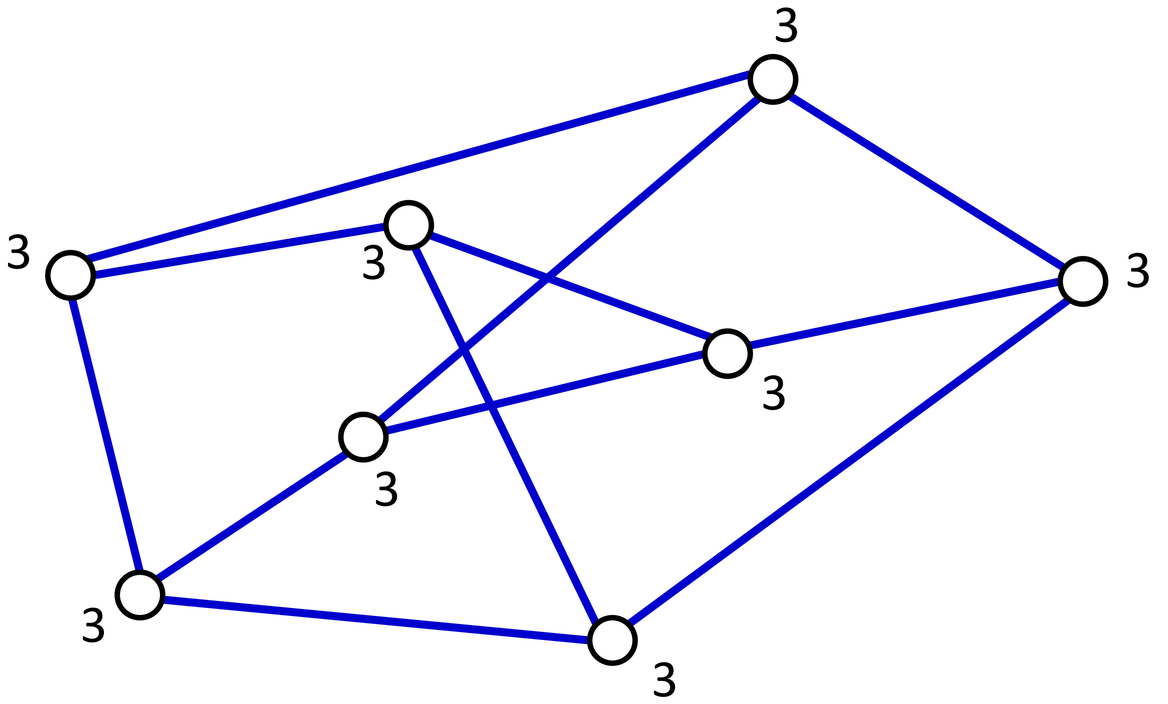
Graph with maximum degree 5



The degree of a graph

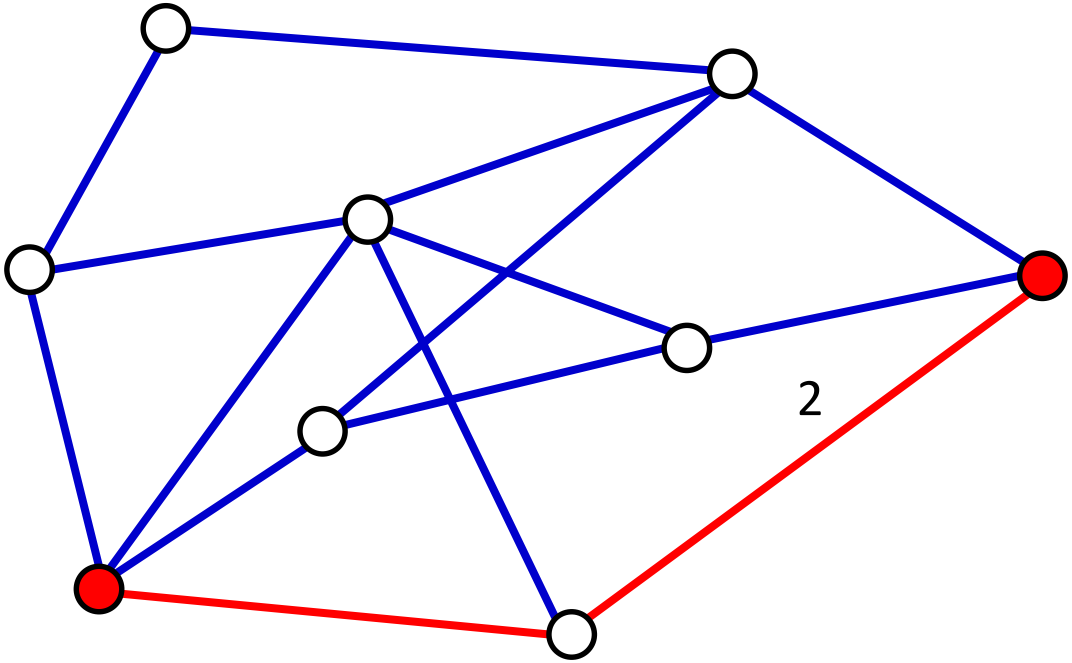
If every vertex has the same degree, then the graph is *regular* and has a *degree.*

Regular graph of degree 3



The diameter of a graph

The *distance* between two vertices is the minimum number of edges in any path between them.



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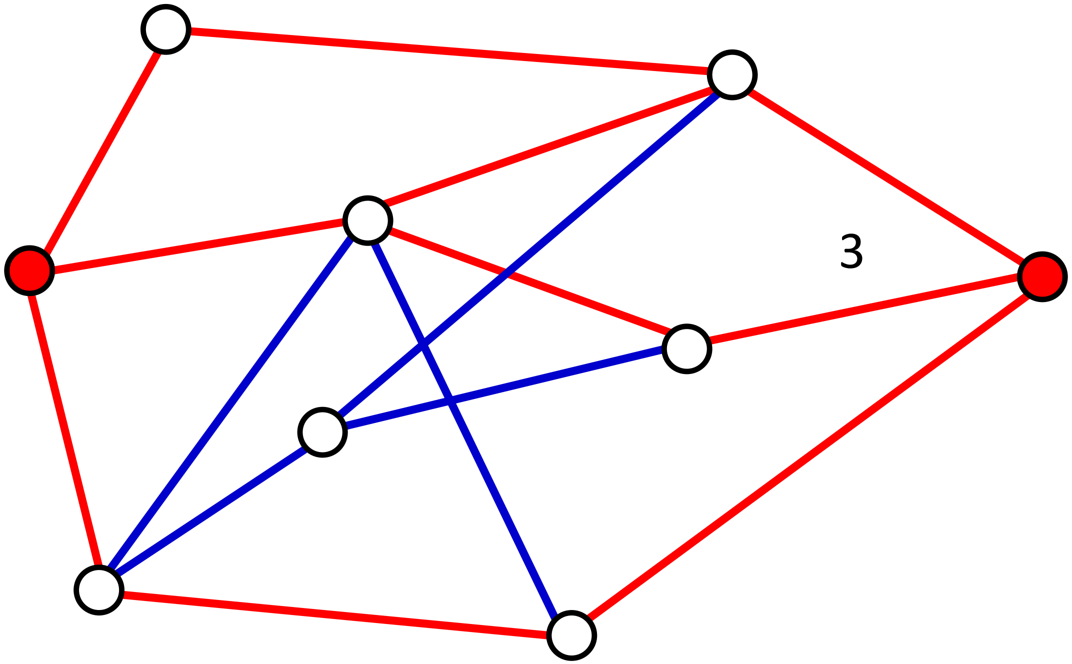
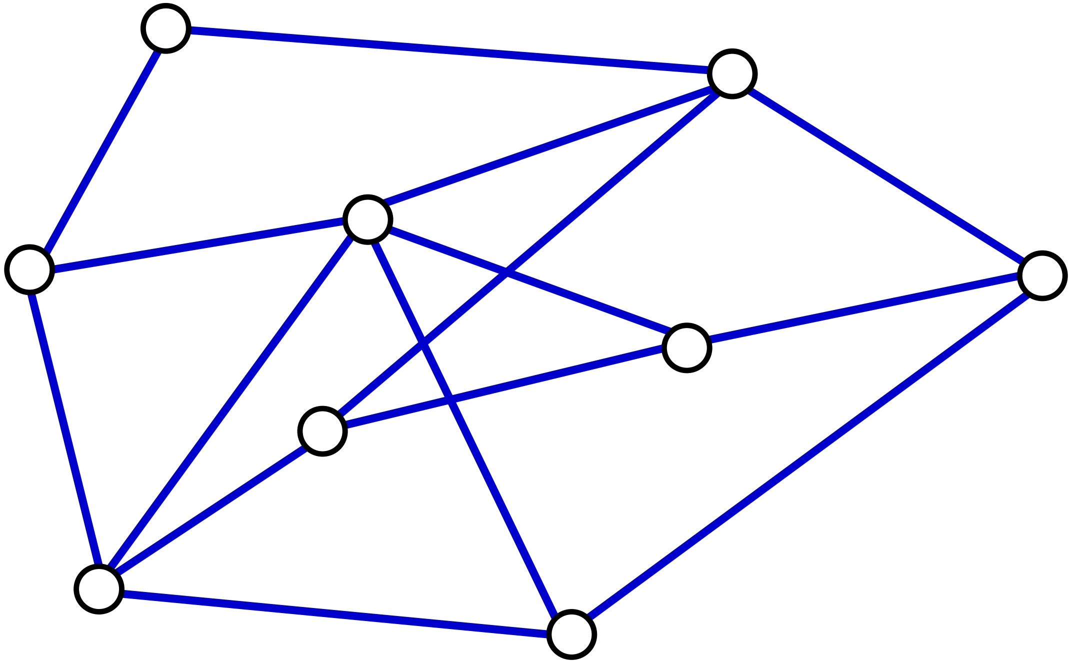


The diameter of a graph

The *diameter* of a graph is the maximum distance between any two vertices.

Graph of diameter 3

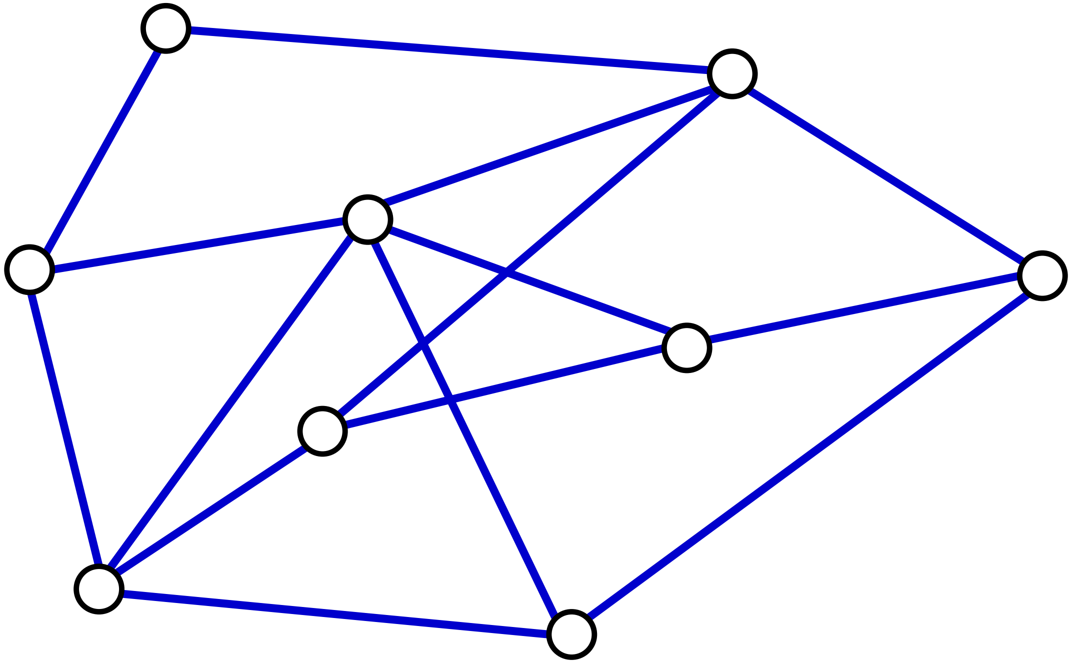


The order, maximum degree and diameter of a graph

The *order* of this graph is 9.

Its *maximum degree* is 5.

Its *diameter* is 3.



The degree-diameter problem

Question:
Given any *maximum degree* and *diameter*,
what is the *largest possible order* of a graph?

The degree-diameter problem seeks to answer that question.

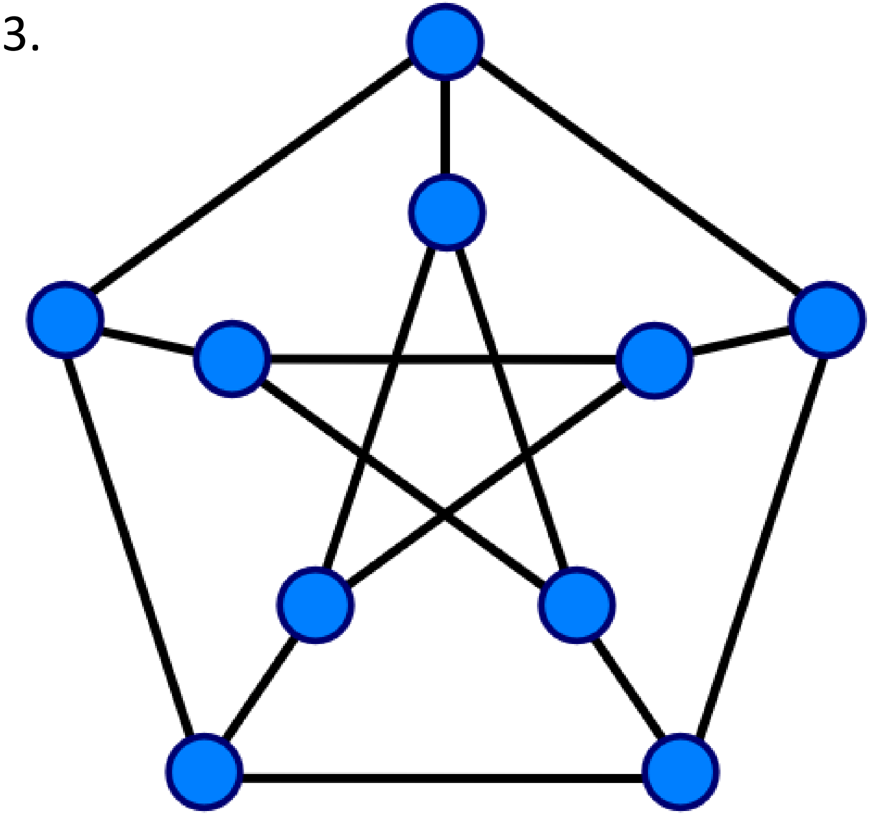
Within this context, such graphs are called *extremal*.

The extremal graph of maximum degree 3 and diameter 2 is the Petersen graph

The *order* of the Petersen graph is 10.

It is regular, with *degree* 3.

Its *diameter* is 2.



The degree-diameter problem is a difficult combinatorial problem

For the general case of undirected graphs, only *seven* graphs with degree greater than 2 are known to be extremal.
Graph order: blue for extremal, (red for largest known)

Degree, d	Diameter, $k = 2$	Diameter, $k = 3$	Diameter, $k = 4$
3:	$10^{(1)}$	20	38
4:	15	(41)	(98)
5:	24	(72)	(212)
6:	32	(111)	(390)
7:	$50^{(2)}$	(168)	(672)

(1) Petersen graph
(2) Hoffman-Singleton graph

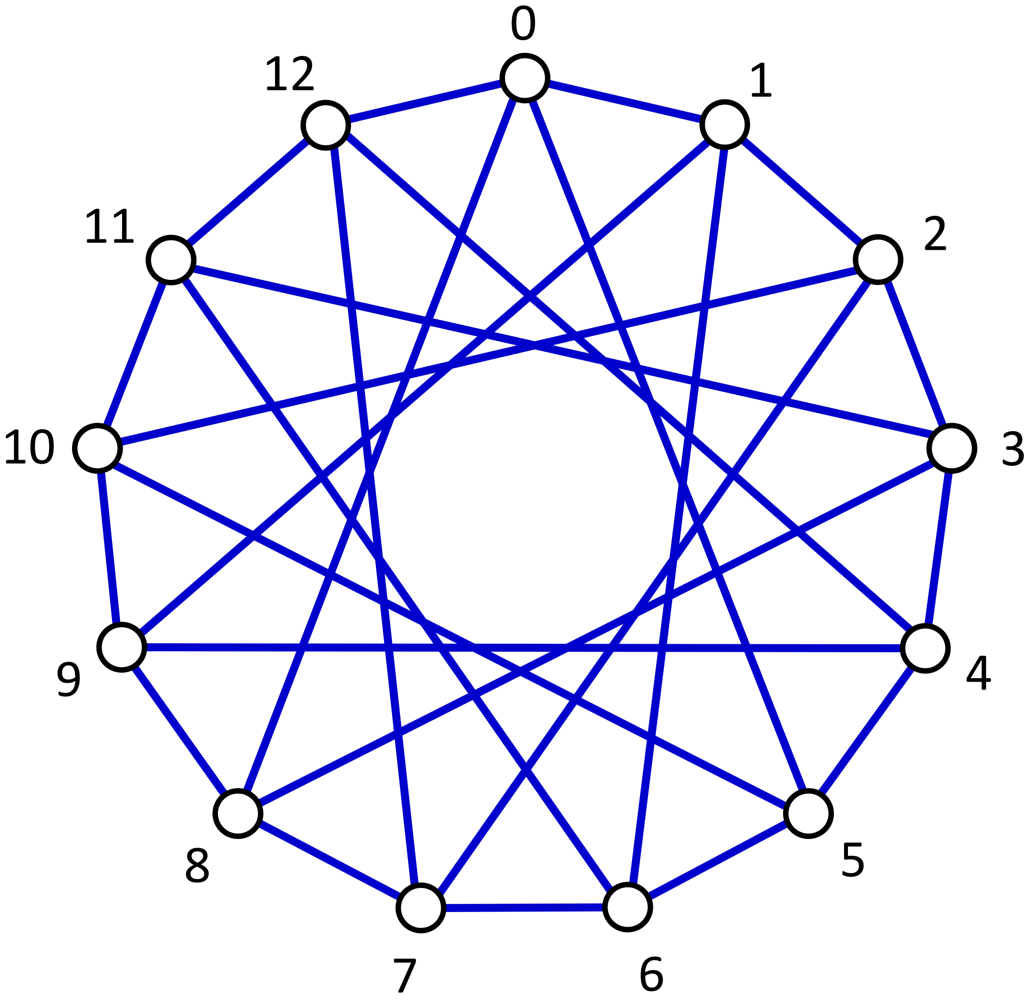
Many more extremal undirected circulant graphs are known because they have a high degree of structure

Graph order: blue for extremal, (red for largest known), for degree d and diameter k

d	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$	Max
3:	8	12	16	20	24	28	32	36	40	All
4:	13	25	41	61	85	113	145	181	221	All
5:	16	36	64	100	144	196	256	324	400	All
6:	21	55	117	203	333	515	737	1027	1393	18
7:	26	76	160	308	536	828	1232	1764	2392	10
8:	35	104	248	528	984	1712	(2768)	(4280)	(6320)	7
9:	42	130	320	700	1416	(2548)	(4304)	(6804)	(10320)	6
10:	51	177	457	1099	(2380)	(4551)	(8288)	(14099)	(22805)	5
11:	56	210	576	(1428)	(3200)	(6652)	(12416)	(21572)	(35880)	4
Max	23	15	11	10	9	8	7	7	7	

Max: Extremal graphs are known up to this limit

An undirected circulant graph of order 13 and degree 4



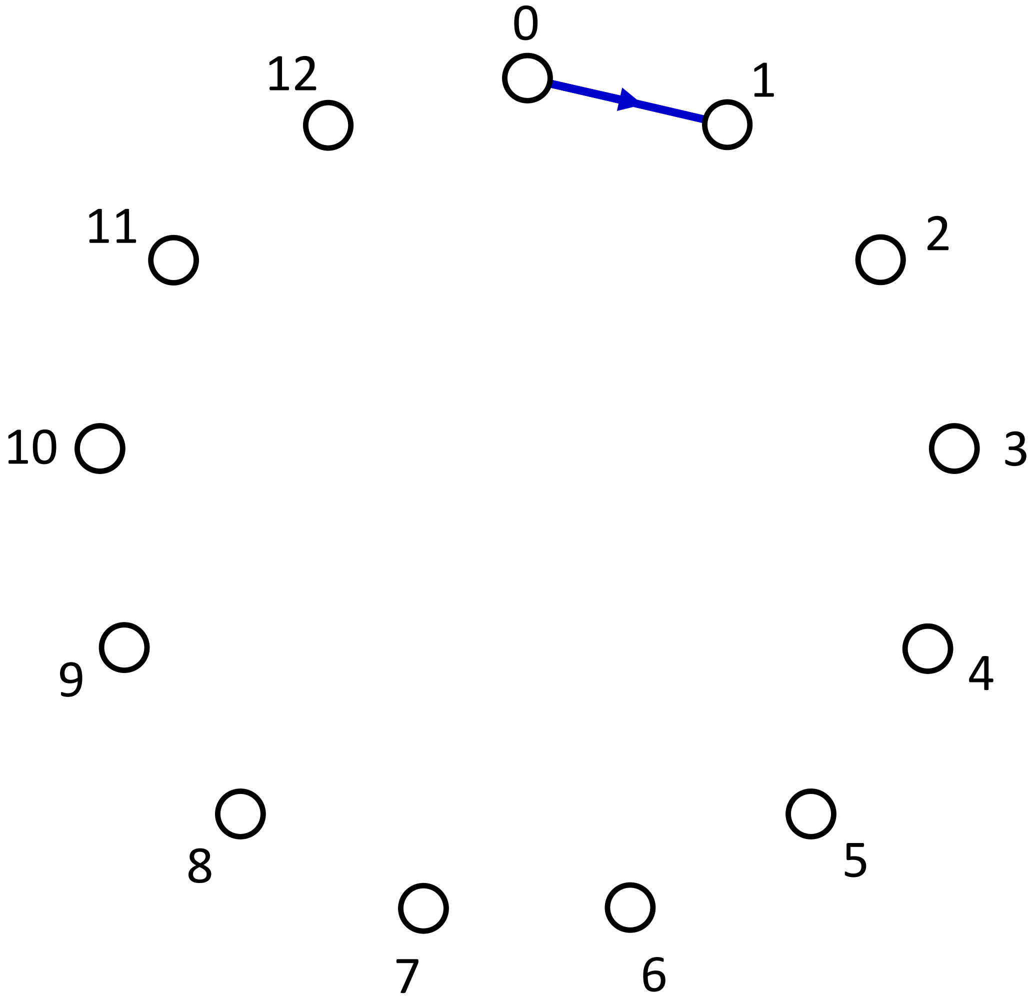
Undirected circulant graph

- Vertex transitive
- Circular symmetry
- Reflexive symmetry

Cayley graph of a cyclic group with self-inverse connection set

- Group $\mathbb{Z}_{13}$
- Order 13
- Degree 4

An undirected circulant graph of order 13 and degree 4



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- Connection element: $+1$

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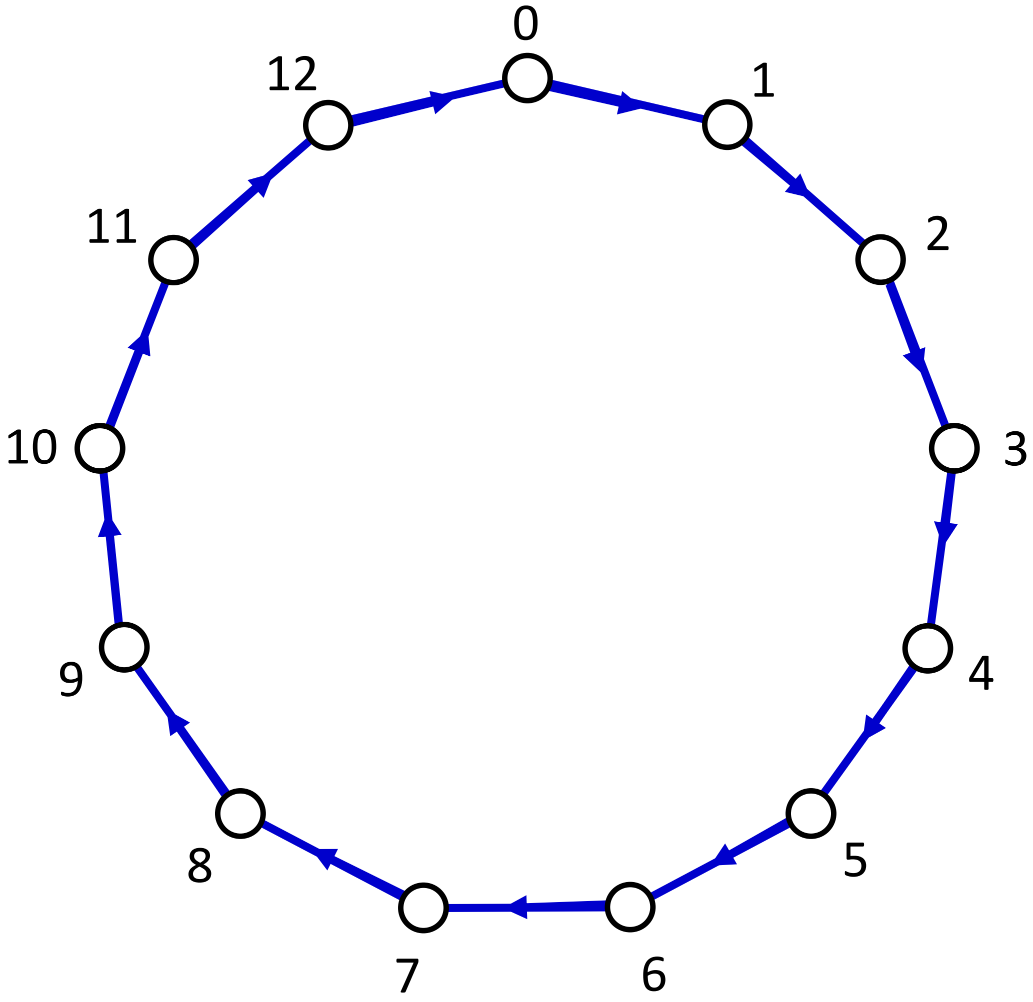

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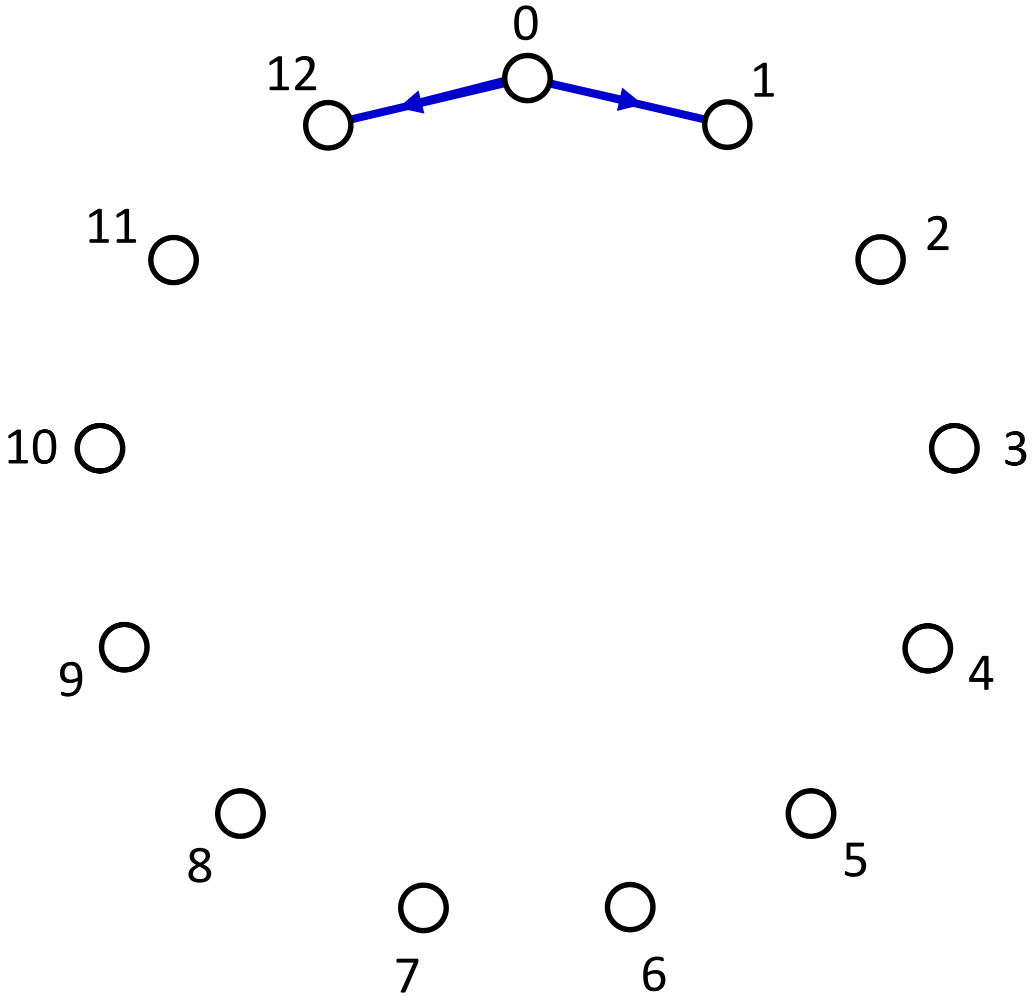

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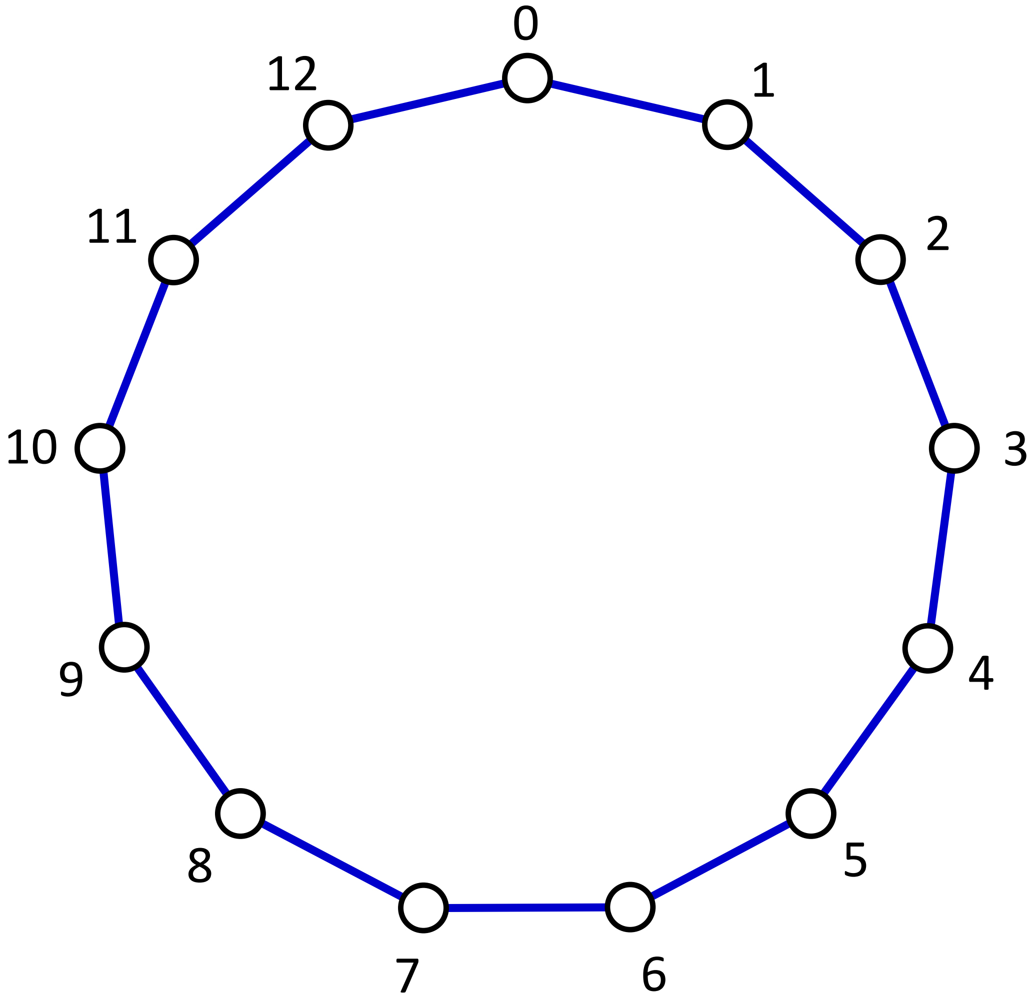

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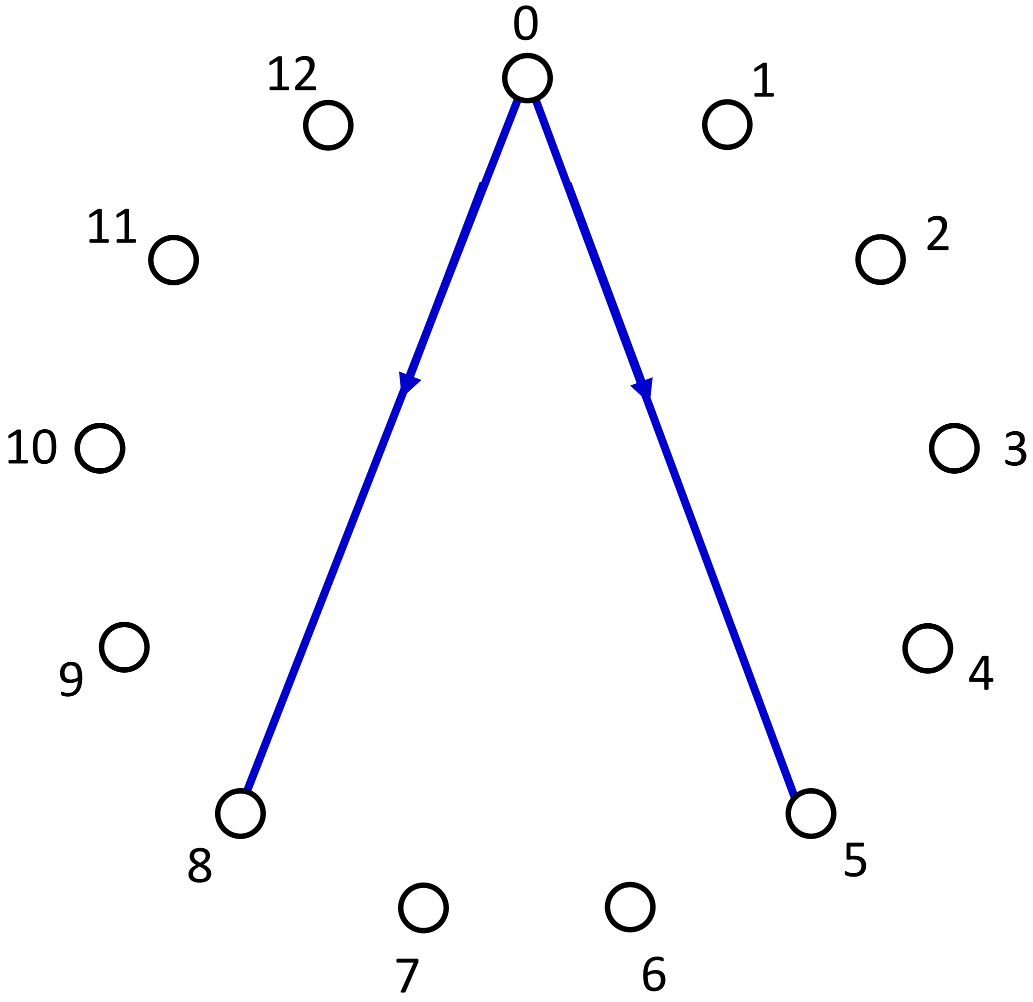

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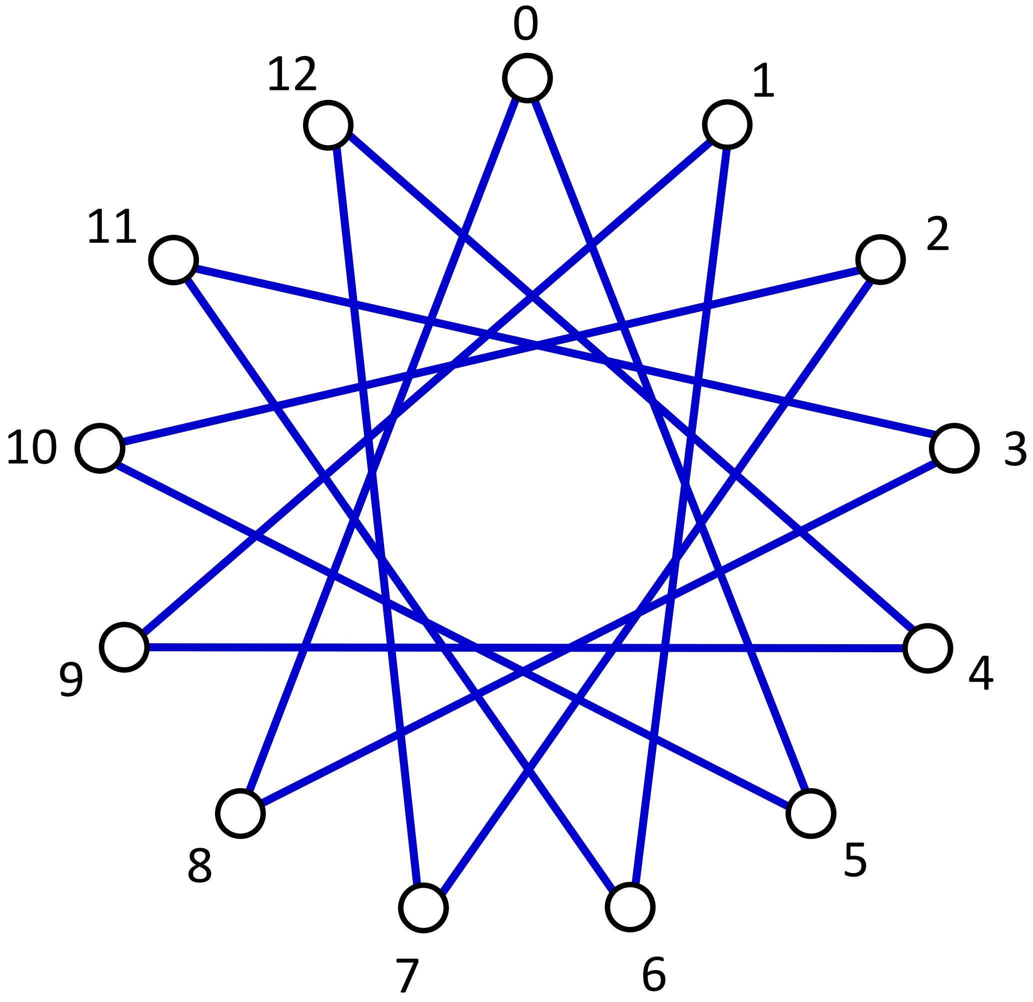

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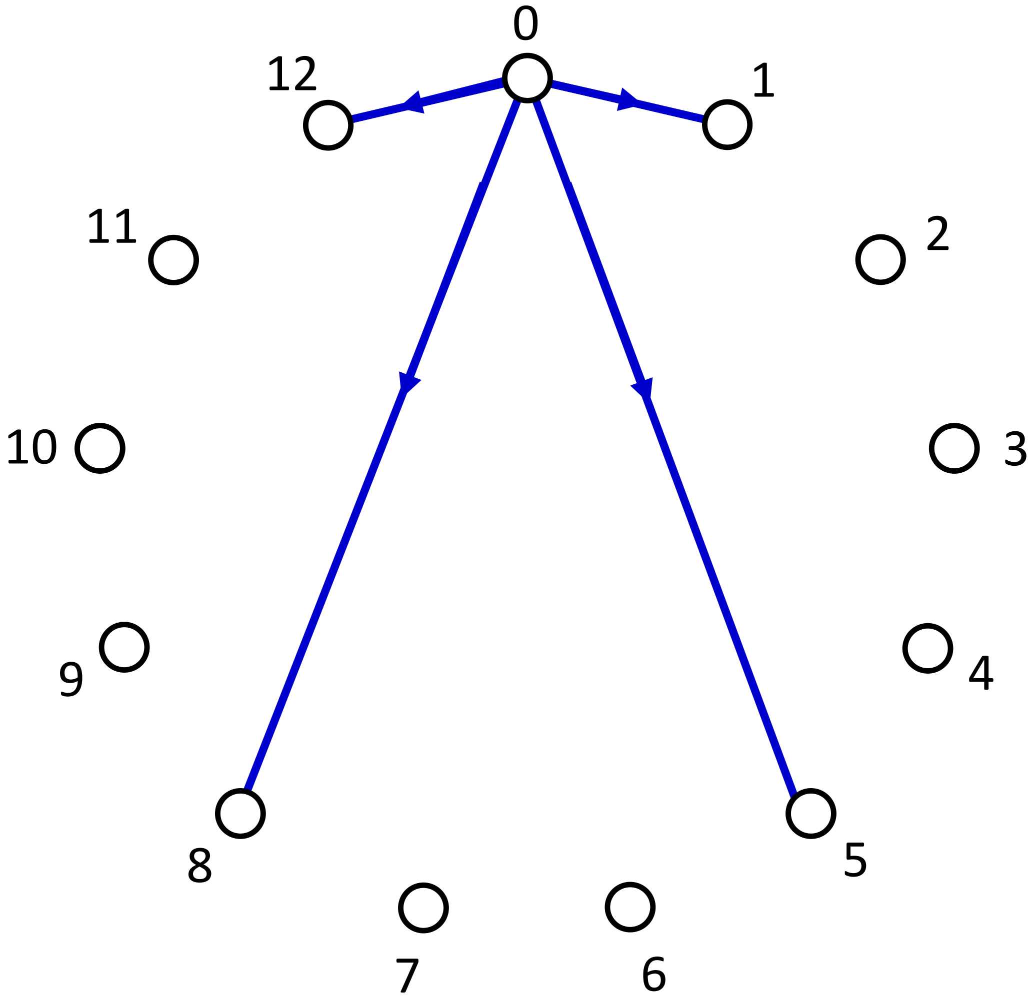

Undirected circulant graph

- Vertex transitive
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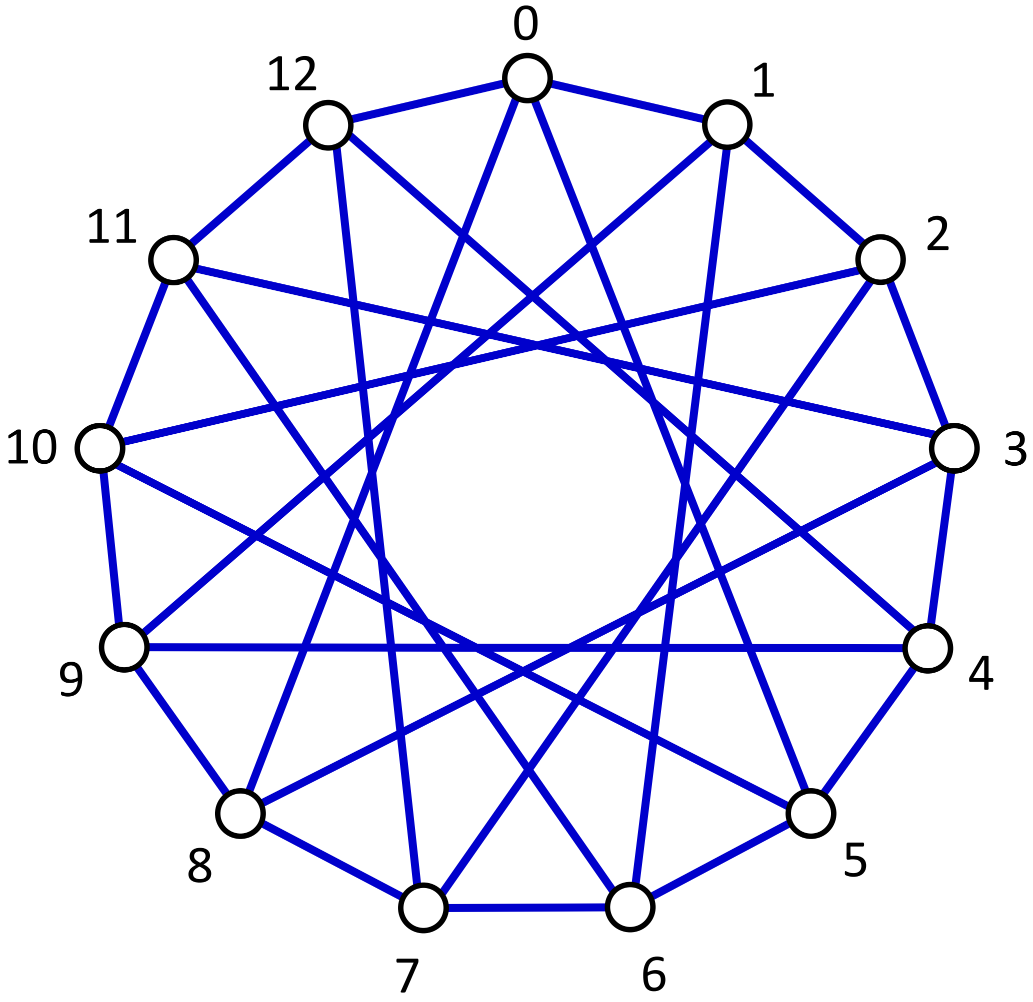
Cayley graph of a cyclic group with self-inverse connection set

- Group $\mathbb{Z}_{13}$
- Order 13
- Degree 4
- Connection set (self-inverse) $\{1,5,8,12\} = \{\pm 1, \pm 5\}$
- Generating set $\{1,5\}$
- Dimension 2

An undirected circulant graph of order 13 and degree 4



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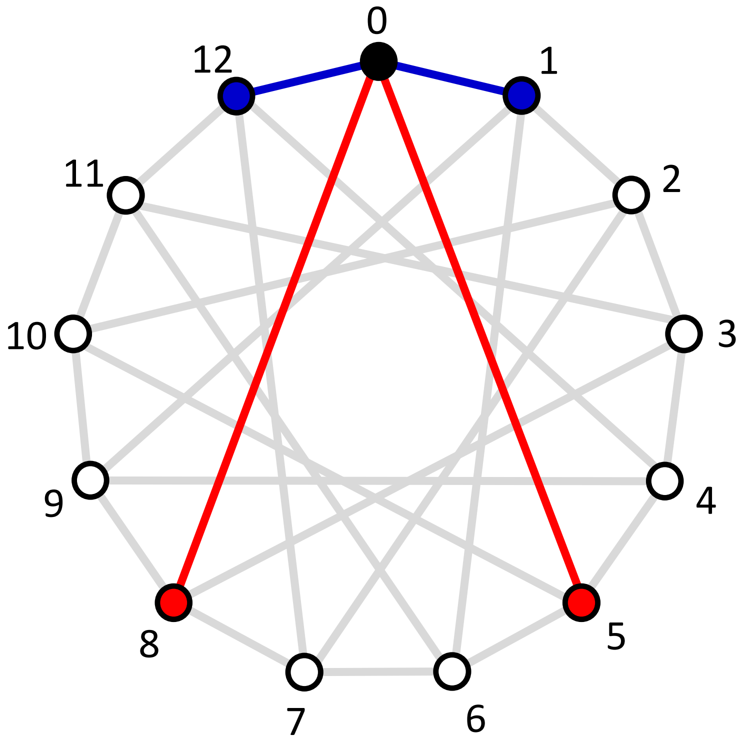
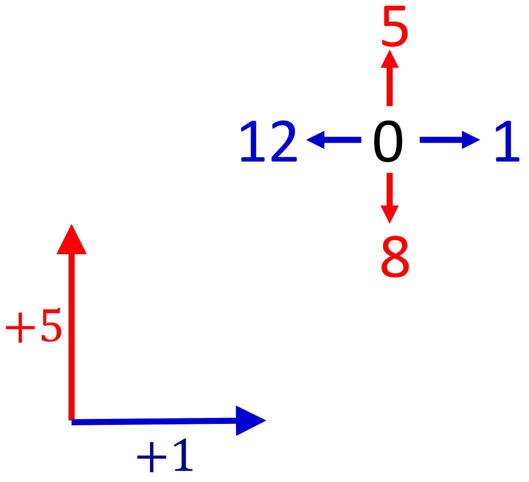
Relating a circulant graph of order 13 and degree 4 to a Lee sphere in $\mathbb{Z}^2$

- We measure distance in $\mathbb{Z}^2$ by the Manhattan metric (rook moves)
- The vertex 0 is arbitrary because circulant graphs are vertex transitive





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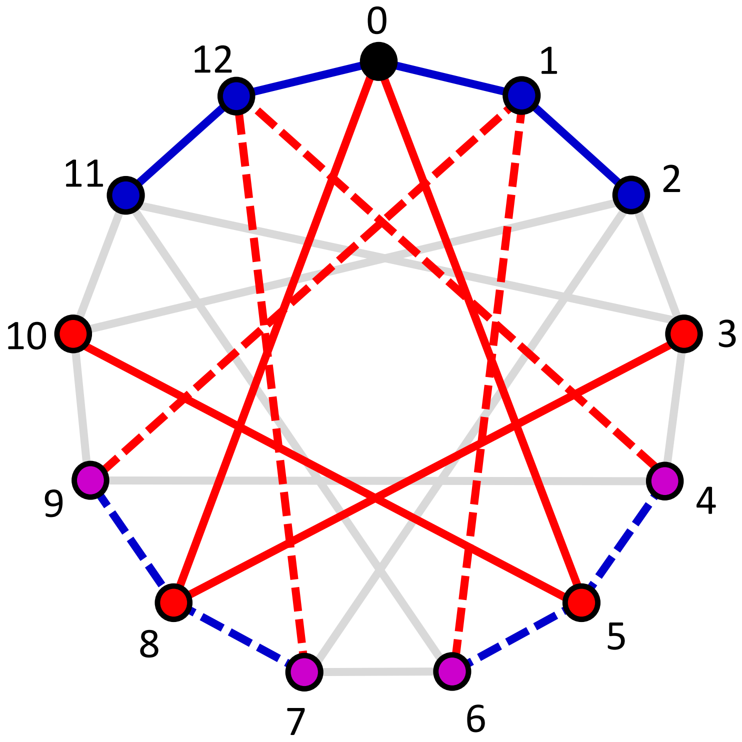
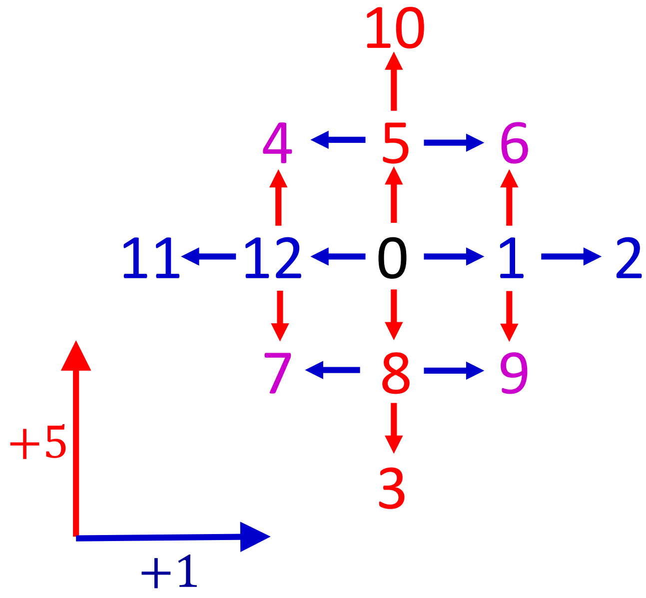




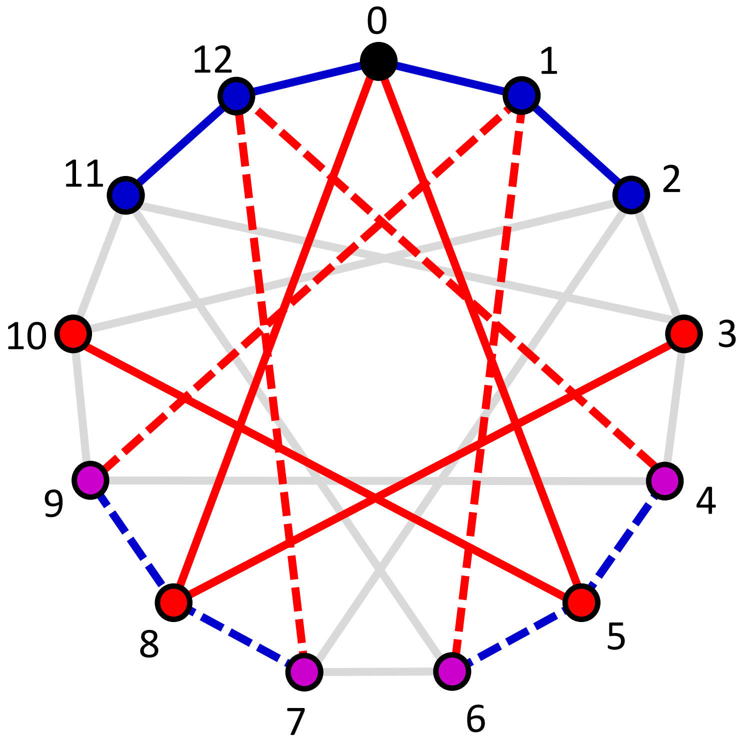
Relating a circulant graph of order 13 and degree 4 to a Lee sphere in $\mathbb{Z}^2$

Graph $\Gamma = Circ(\mathbb{Z}_{13}, \{1,5\})$

Dimension $f = 2$

Degree $d = 4$

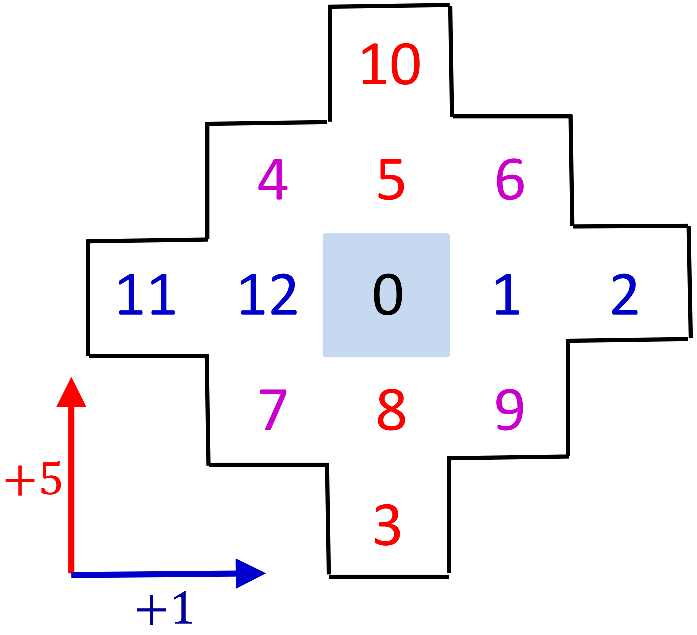
Diameter $k = 2$



Lee sphere in $\mathbb{Z}^2$, centre 0, basis values 1 & 5

Dimension $f = 2$

Radius $k = 2$

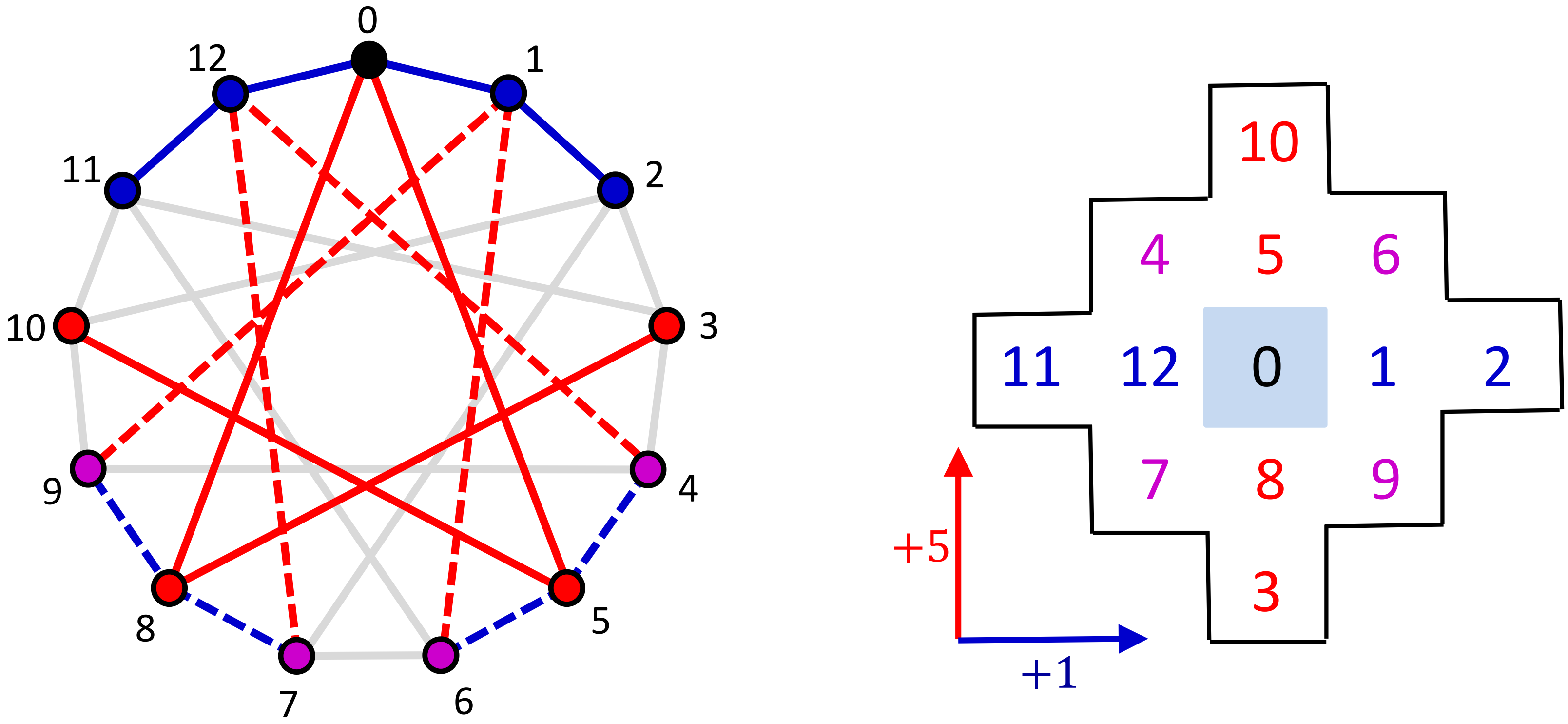


Relating a circulant graph of order 13 and degree 4 to a Lee sphere in $\mathbb{Z}^2$

The volume of the Lee sphere of radius 2 in $\mathbb{Z}^2$ is 13.

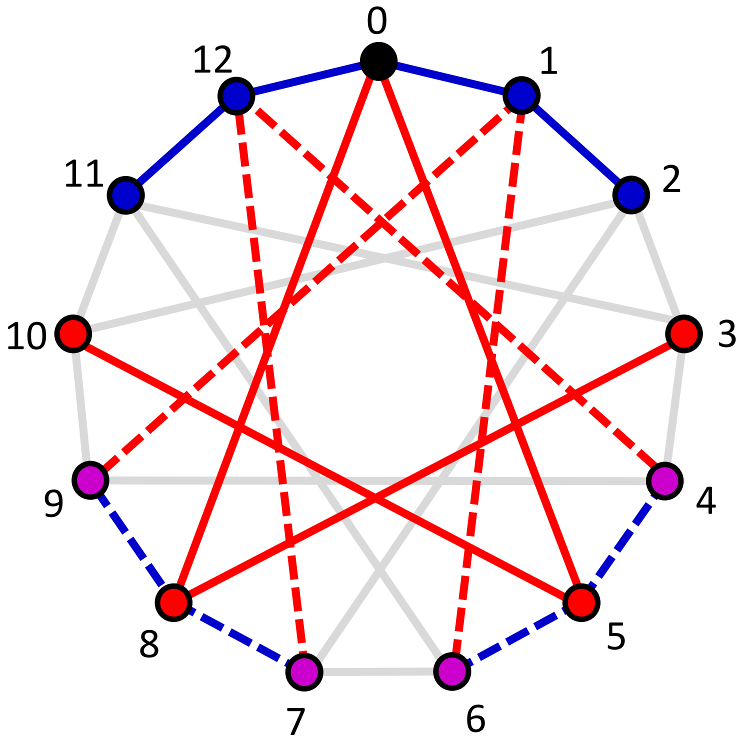
So any degree 4 circulant graph with order greater than 13 must map to a Lee sphere with radius greater than 2.

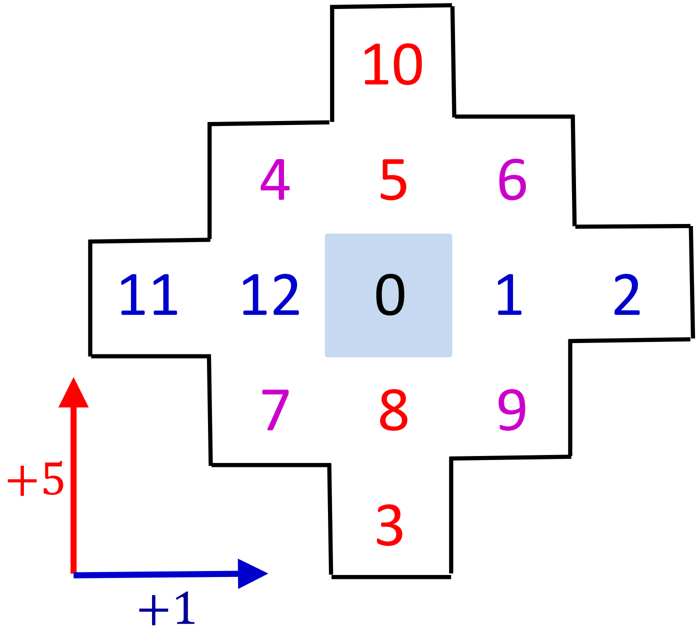
So 13 is an upper bound for the order of a degree 4 circulant graph of diameter 2.



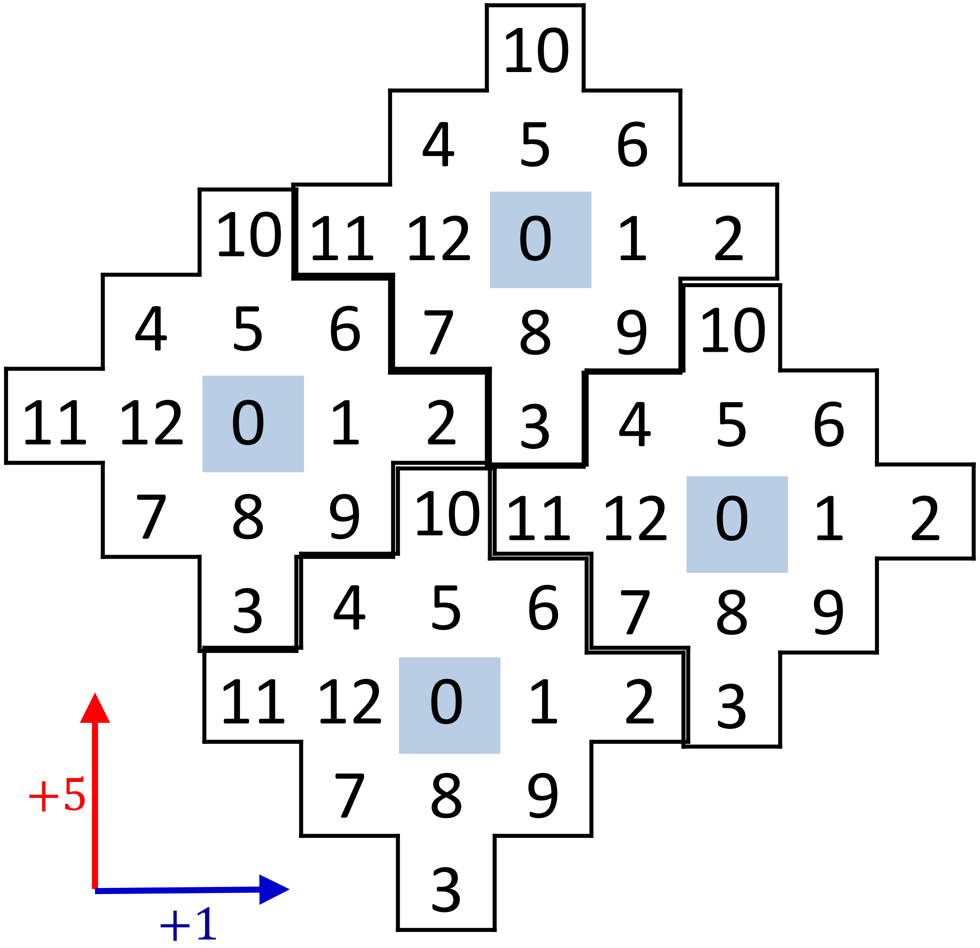
Relating a circulant graph of order 13 and degree 4 to a Lee sphere in $\mathbb{Z}^2$

More generally, the volume of the Lee sphere of radius k in $\mathbb{Z}^f$ is an upper bound for the order of a circulant graph of degree $d = 2f$ and diameter k.

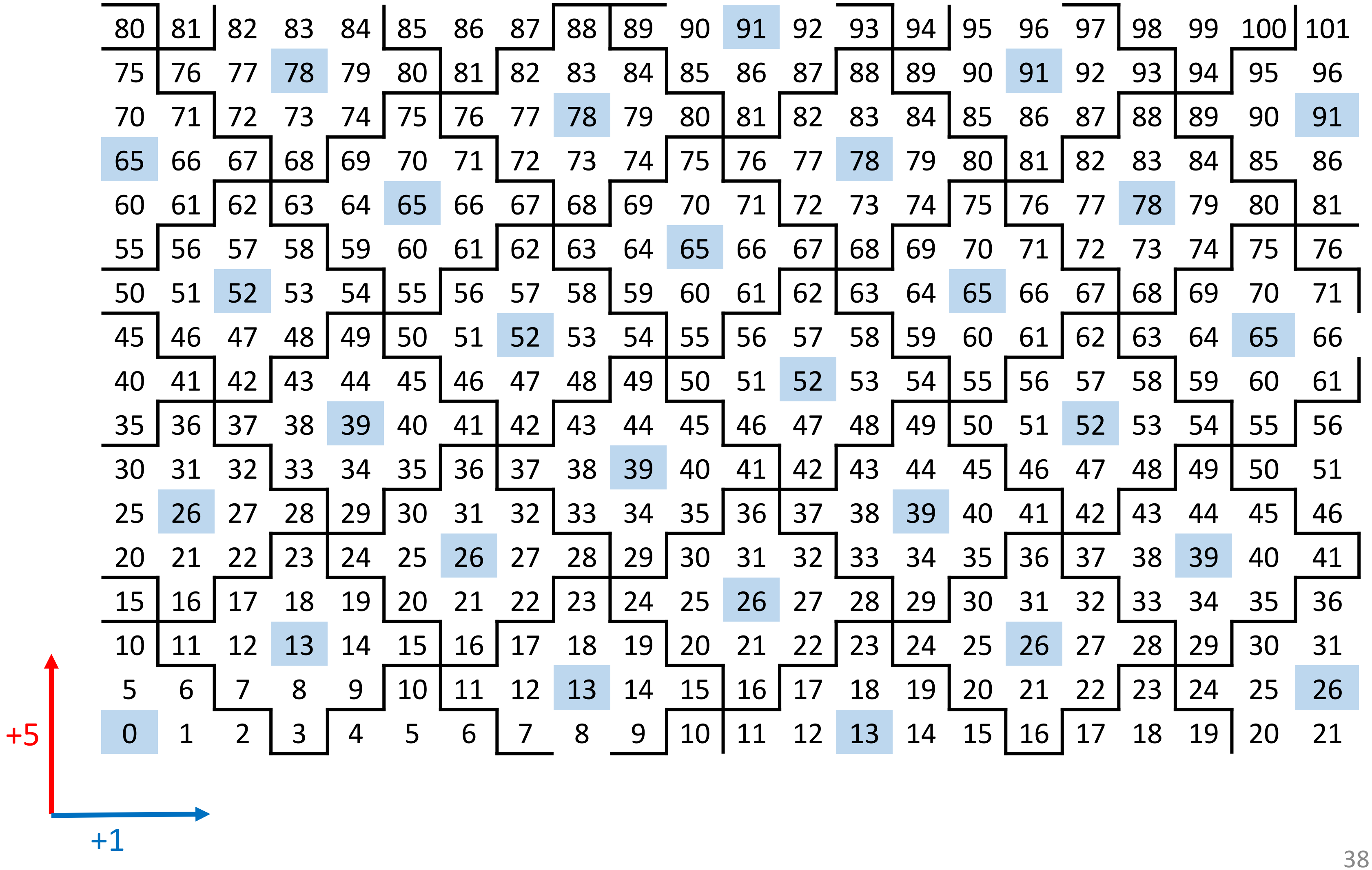




These Lee spheres form a perfect lattice tiling of $\mathbb{Z}^2$ (modulo 13)

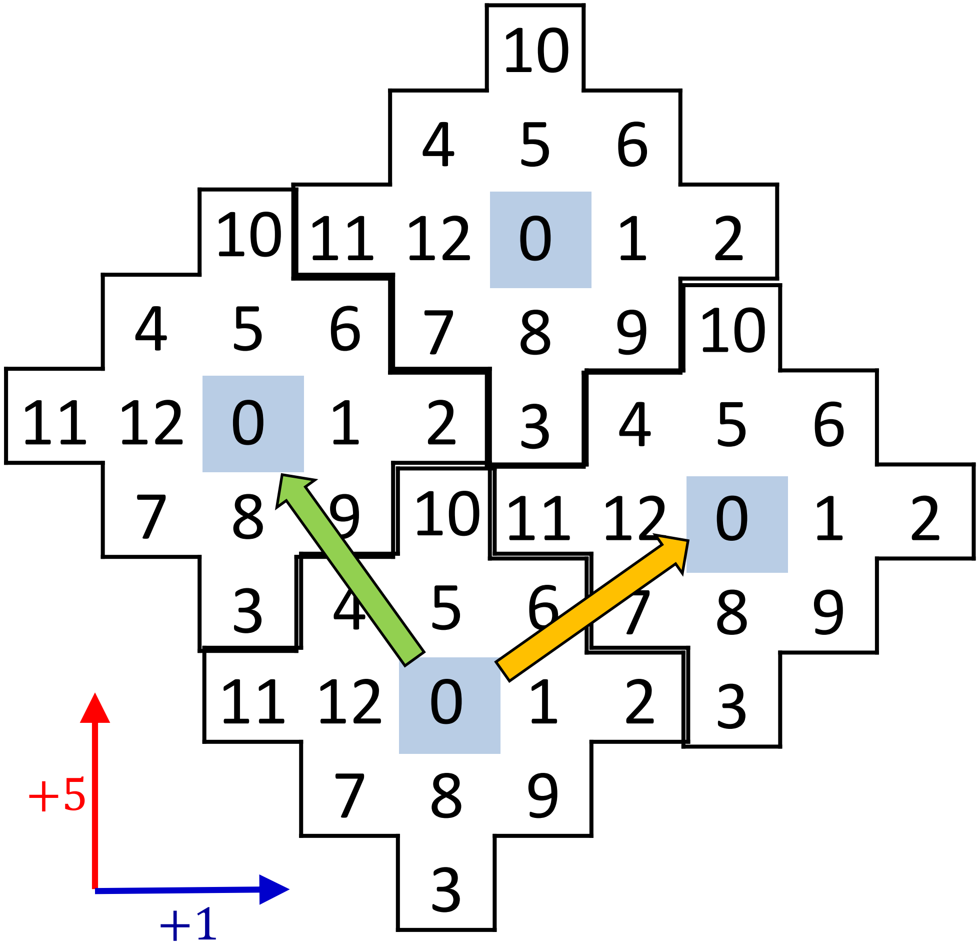


Degree 4 (dimension 2), diameter 2, order 13: Perfect lattice tiling of $\mathbb{Z}^2$ with generating set $\{1, 5\}$



80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101
75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96
70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91
65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86
60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81
55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76
50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71
45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66
40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56
30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

A set of generating vectors for the extremal degree 4 diameter 2 lattice



Lattice generating vectors for diameter $k = 2$

➡ $V_1 = (3, 2)$

➡ $V_2 = (-2, 3)$

So we have the lattice generator matrix (LGM):

$$M = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}.$$

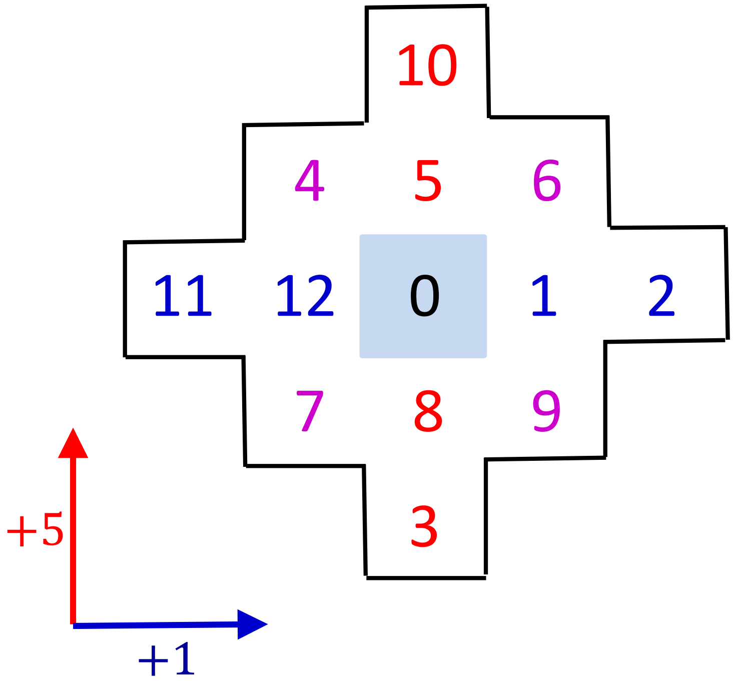
Area of the lattice unit cell, $|\det M| = 13$, the order of the graph.

Area of Lee sphere is also 13. So the covering density is 1 (perfect)

Not every circulant graph of order 13 and degree 4 has diameter 2

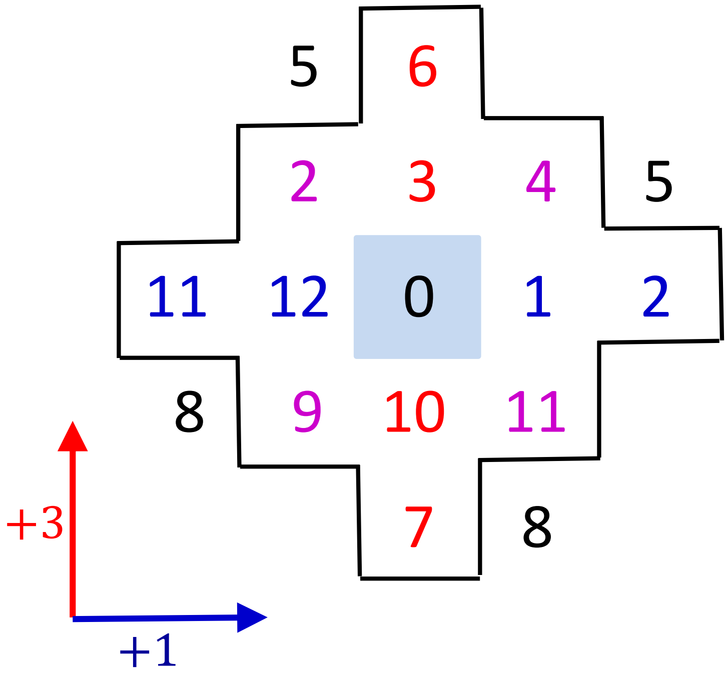
Graph $\Gamma = Circ(\mathbb{Z}_{13}, \{1, 5\})$
Diameter $k = 2$

Lee sphere in $\mathbb{Z}^2$, basis values 1 & 5
Radius $k = 2$
Contains every vertex exactly once

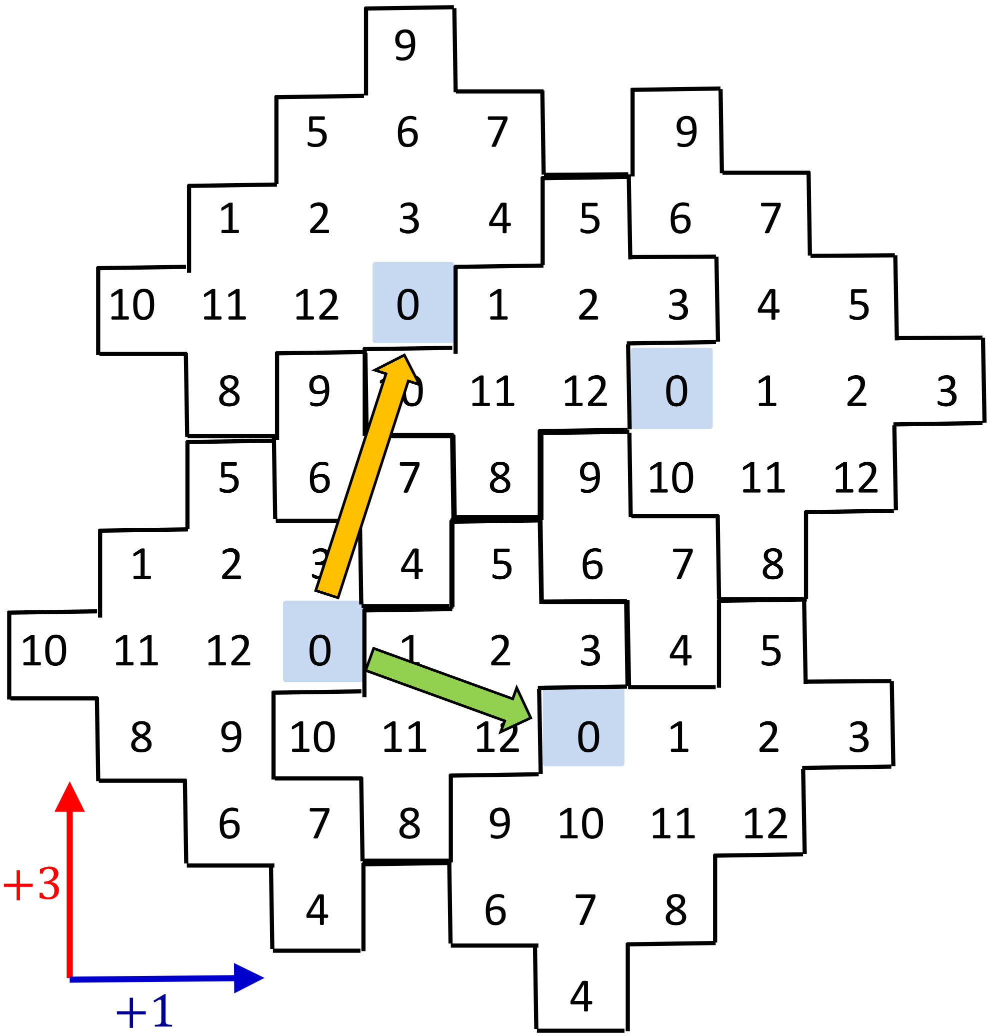


Graph $\Lambda = Circ(\mathbb{Z}_{13}, \{1, 3\})$
Diameter $k = 3$

Lee sphere in $\mathbb{Z}^2$, basis values 1 & 3
Radius $k = 2$
Duplicates 2 & 11, and omits 5 & 8



$Circ(\mathbb{Z}_{13},\{1,3\})$ gives a lattice covering of $\mathbb{Z}^2$ with Lee spheres of radius 3 that is not a perfect tiling



Lattice generating vectors for diameter $k = 3$

➡ $V_1 = (1, 4)$

➡ $V_2 = (3, -1)$

So we have the lattice generator matrix (LGM):

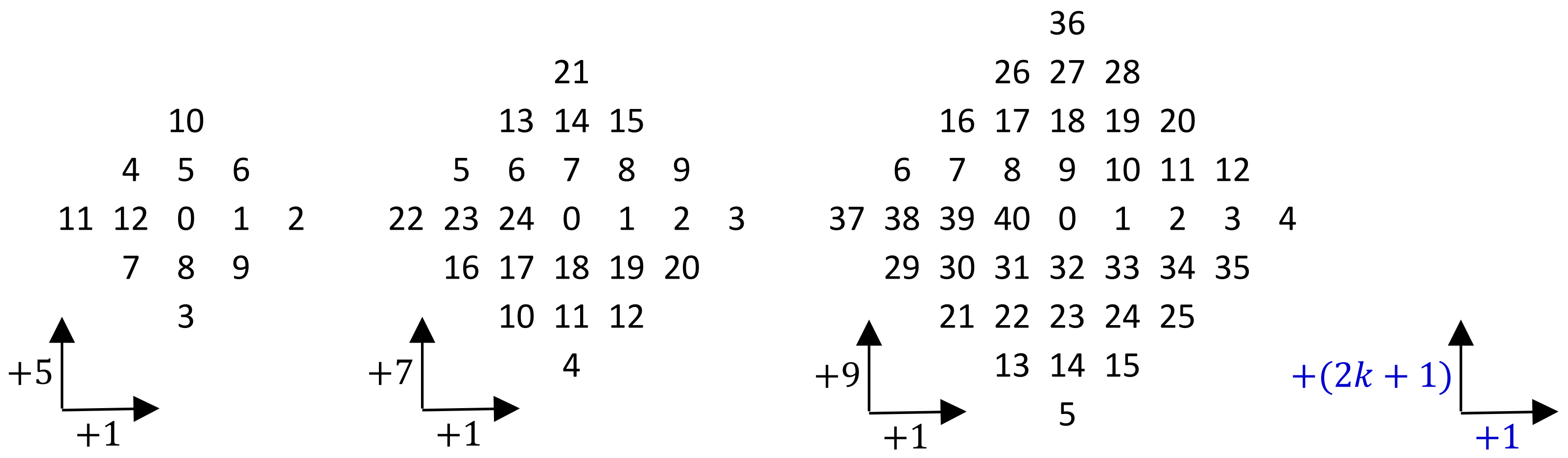
$$M = \begin{pmatrix} 1 & 4 \\ 3 & -1 \end{pmatrix}.$$

Area of the lattice unit cell, $|\det M| = 13$, the order of the graph.

Area of Lee sphere is 25. So the covering density is $\frac{25}{13} \cong 1.9$

The extremal degree 4 circulant graphs of arbitrary diameter form a graph family

Lee spheres of dimension 2 and radius k



$k = 2$

		10		
	4	5	6	
11	12	0	1	2
	7	8	9	
		3		

$k = 3$

			21			
		13	14	15		
	5	6	7	8	9	
22	23	24	0	1	2	3
	16	17	18	19	20	
		10	11	12		
			4			

$k = 4$

				36				
			26	27	28			
		16	17	18	19	20		
	6	7	8	9	10	11	12	
37	38	39	40	0	1	2	3	4
	29	30	31	32	33	34	35	
		21	22	23	24	25		
			13	14	15			
				5				

Any k

Lattice generating matrix

$$\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 5 & 4 \\ -4 & 5 \end{pmatrix} \qquad \begin{pmatrix} k+1 & k \\ -k & k+1 \end{pmatrix}$$

Extremal degree 4 circulant graphs

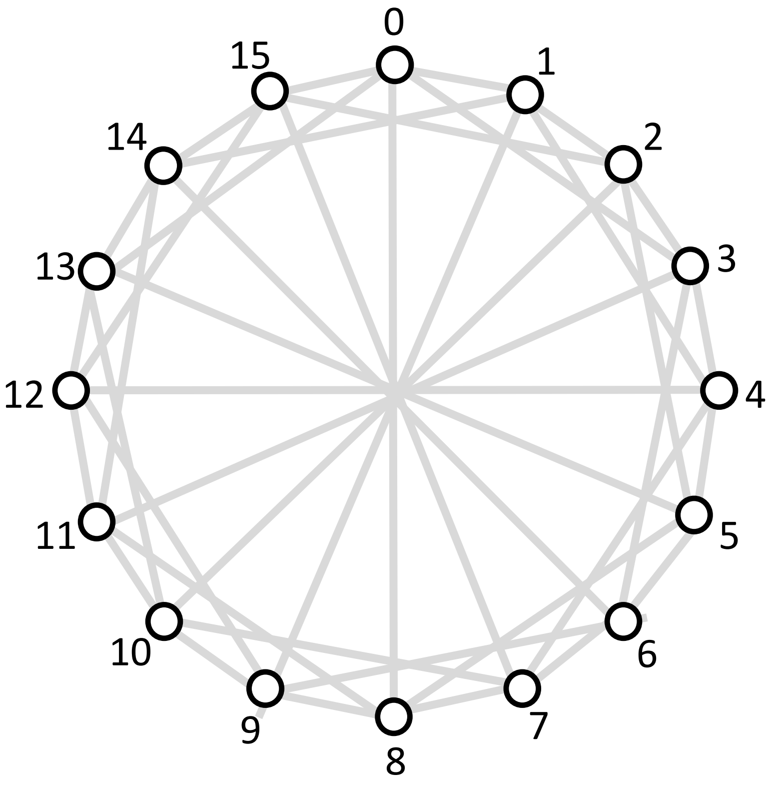
Order:	13	25	41	$2k^2 + 2k + 1$
Gen set:	$\{1, 5\}$	$\{1, 7\}$	$\{1, 9\}$	$\{1, 2k + 1\}$
Diameter:	2	3	4	k

Circulant graphs of odd degree

For undirected circulant graphs, the connection set is self-inverse.

So the degree can only be odd if the connection set includes an involution.

For a graph of order n, the only involution is $n/2$, so that the graph has even order.



Order $n = 16$

Degree $d = 5$

Connection set $\{\pm 1, \pm 3, 8\}$

Generating set $\{1, 3\}$

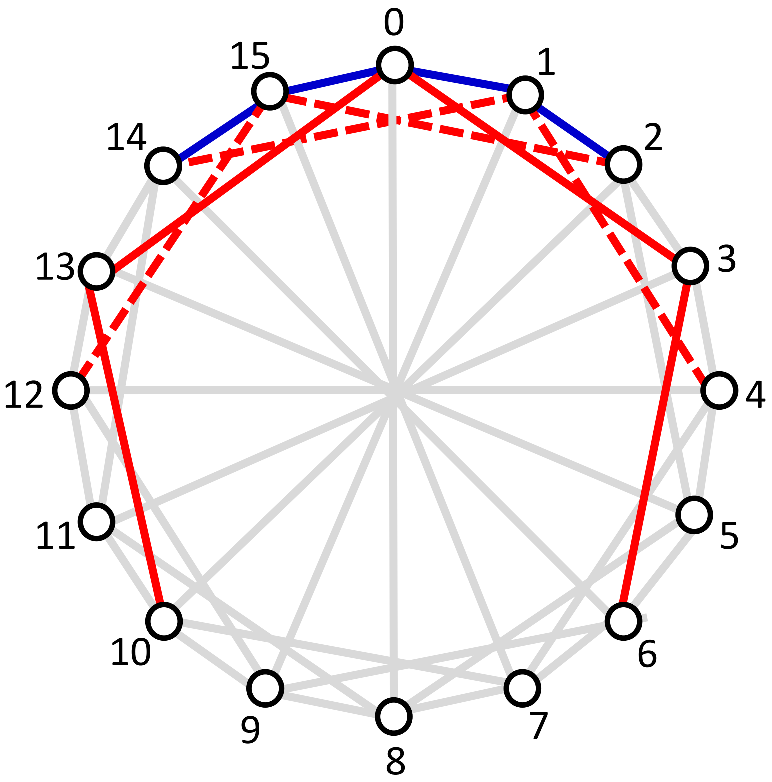
Dimension $f = 2$

Diameter $k = 2$

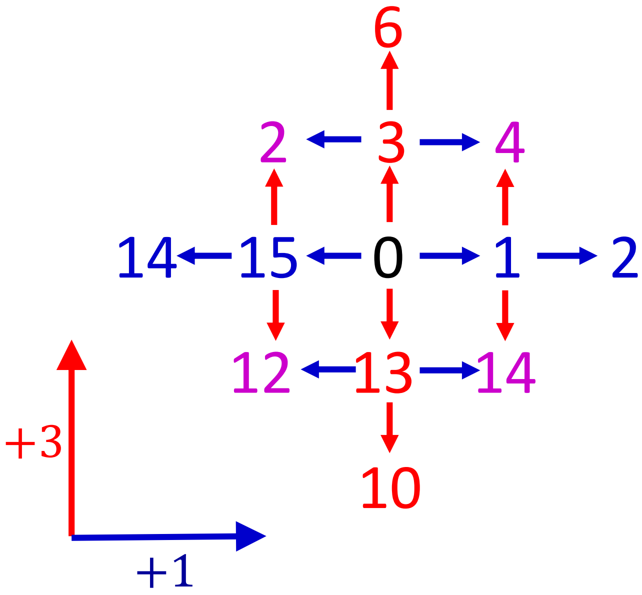
Some shortest paths between pairs of vertices include the involution, others do not.

Shortest paths between vertices that avoid the involution

Shortest paths of length 1 and 2 avoiding the involution correspond to a Lee sphere of radius 2, centred on arbitrary vertex 0.

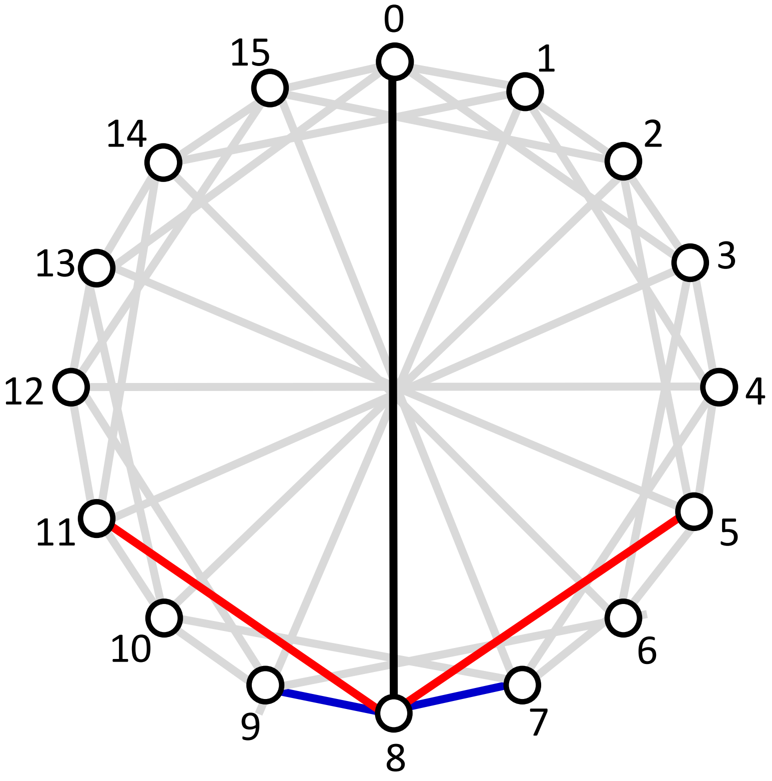


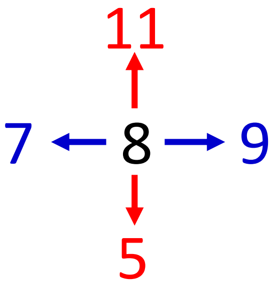
We note that vertices 2 and 14 occur twice in the Lee sphere

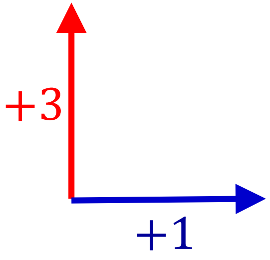


Shortest paths between vertices that include the involution

Shortest paths of length 1 and 2 including the involution correspond to a Lee sphere of radius 1, centred on vertex 8, because one edge is fixed by the involution.

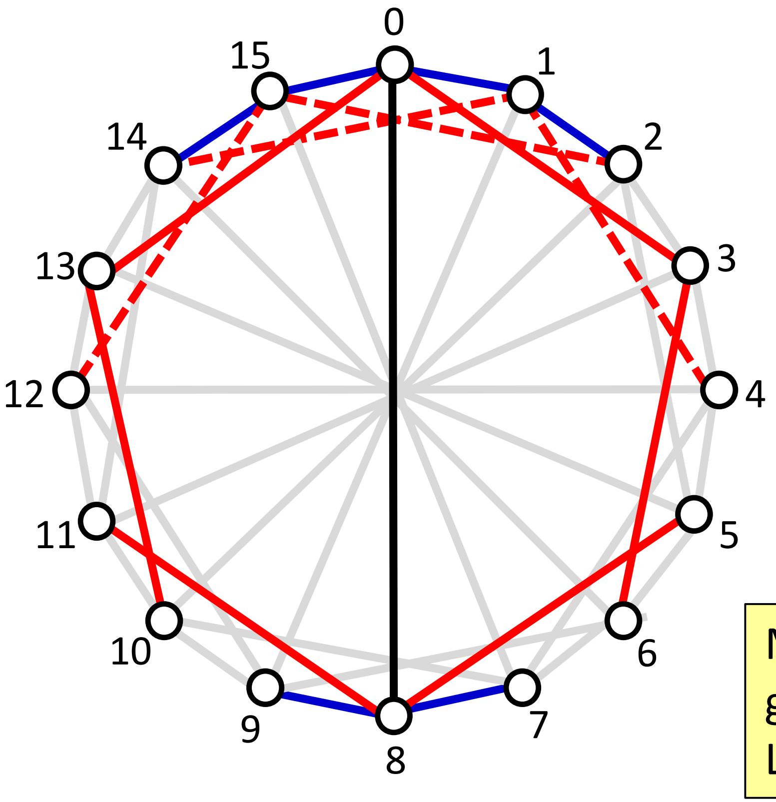


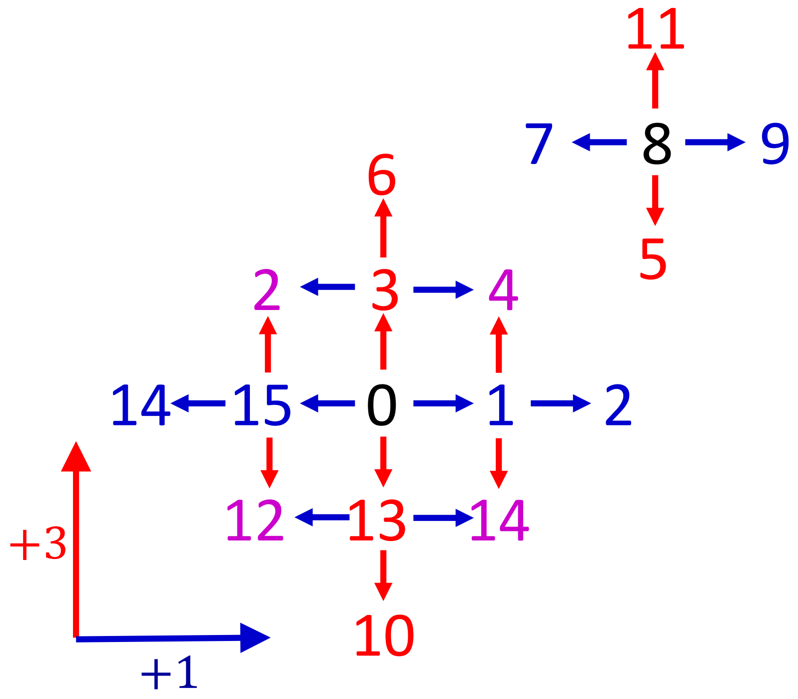




Shortest paths between vertices that both include and avoid the involution

Shortest paths of length 1 and 2 that include or avoid the involution correspond to the combination of a Lee sphere of radius 2, centred on vertex 0, and one of radius 1, centred on vertex 8.

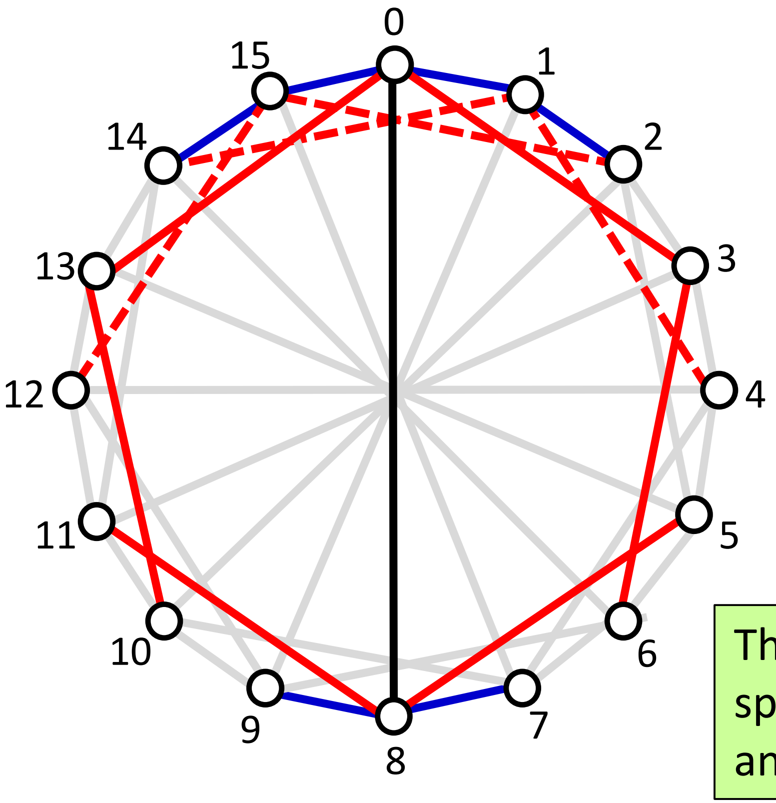


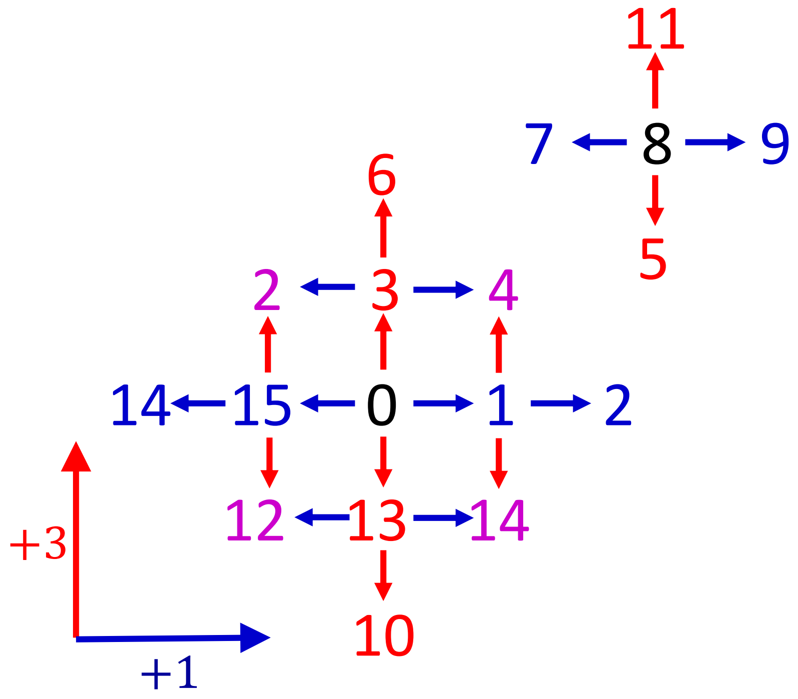


More generally, an odd-degree circulant graph of diameter k corresponds to two Lee spheres, of radii k and $k-1$.

Shortest paths between vertices that both include and avoid the involution

Shortest paths of length 1 and 2 that include or avoid the involution correspond to the combination of a Lee sphere of radius 2, centred on vertex 0, and one of radius 1, centred on vertex 8.





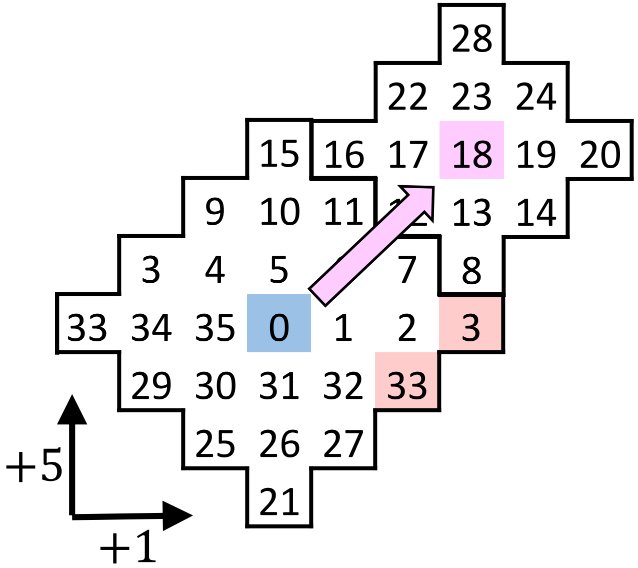
The sum of the volumes of these two Lee spheres is an upper bound for the order of an odd-degree circulant graph of diameter k

The Lee spheres for the extremal degree 5 circulant graph of diameter 3 and order 36

Vertices lying a maximum distance of 3 from vertex 0 with a path avoiding the involution are represented in a Lee sphere of radius 3 centred on 0.

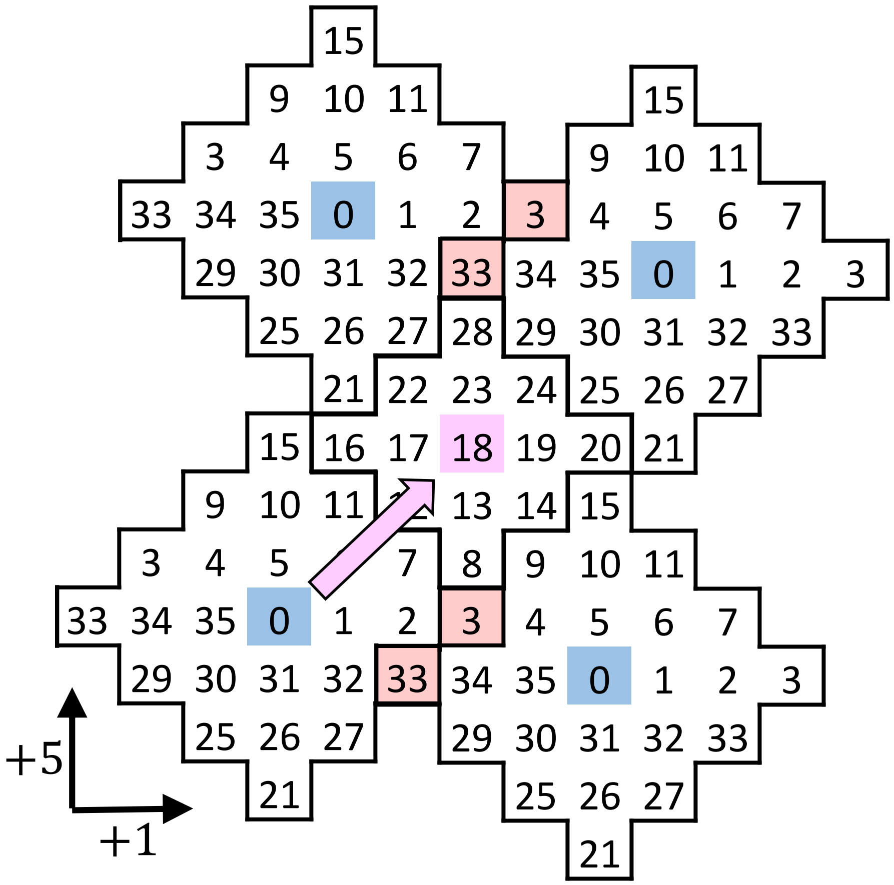
As the order is 36, the involutory connection element joins vertex 0 to vertex 18.

Vertices lying a maximum distance of 3 from vertex 0 with a path including the involution are represented in a Lee sphere of radius 2 centred on 18.

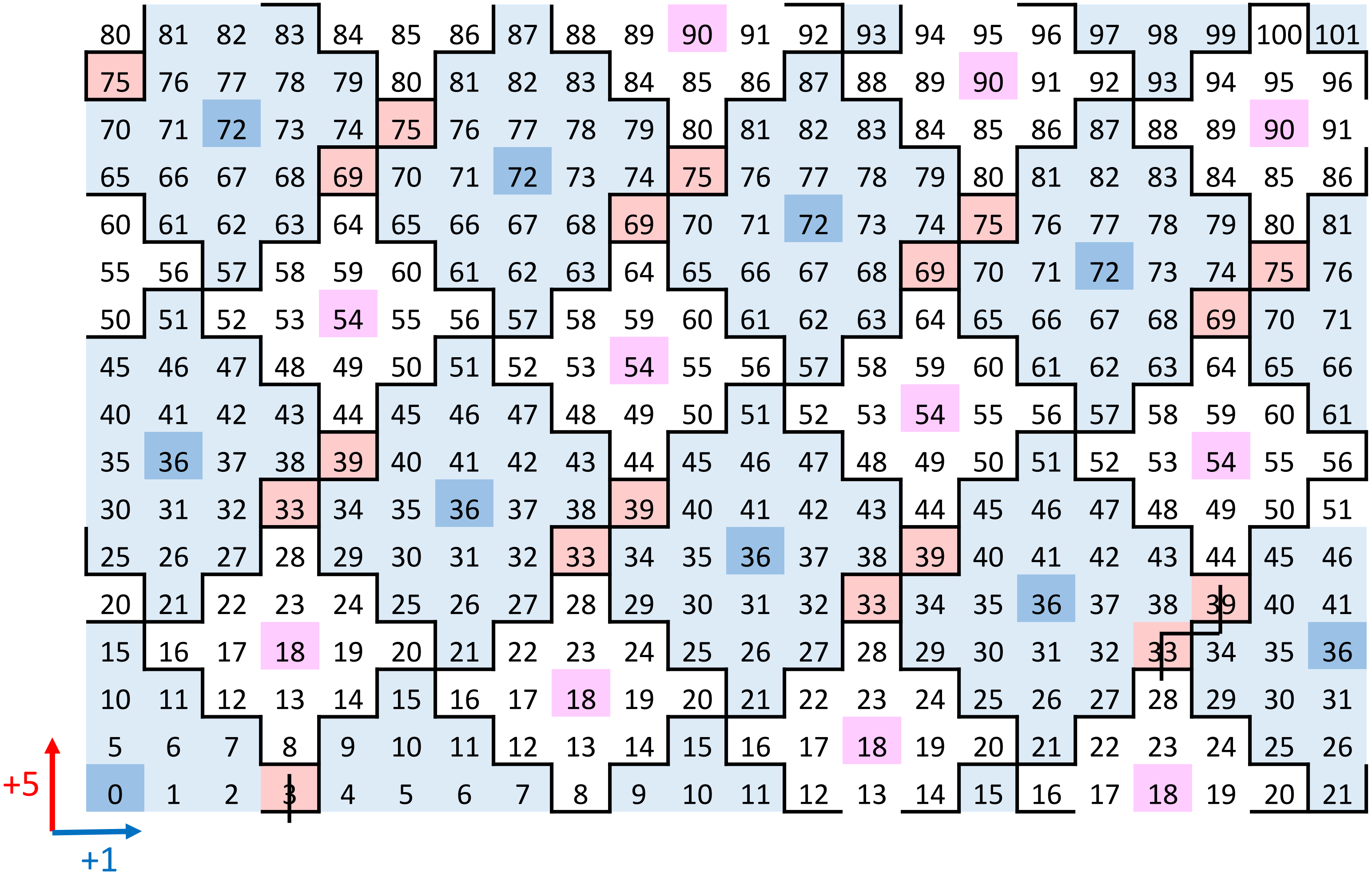


						28		
					22	23	24	
			15	16	17	18	19	20
		9	10	11	12	13	14	
	3	4	5		7	8		
33	34	35	0	1	2	3		
	29	30	31	32	33			
		25	26	27				
			21					

The Lee spheres for the extremal degree 5 circulant graph of diameter 3 and order 36



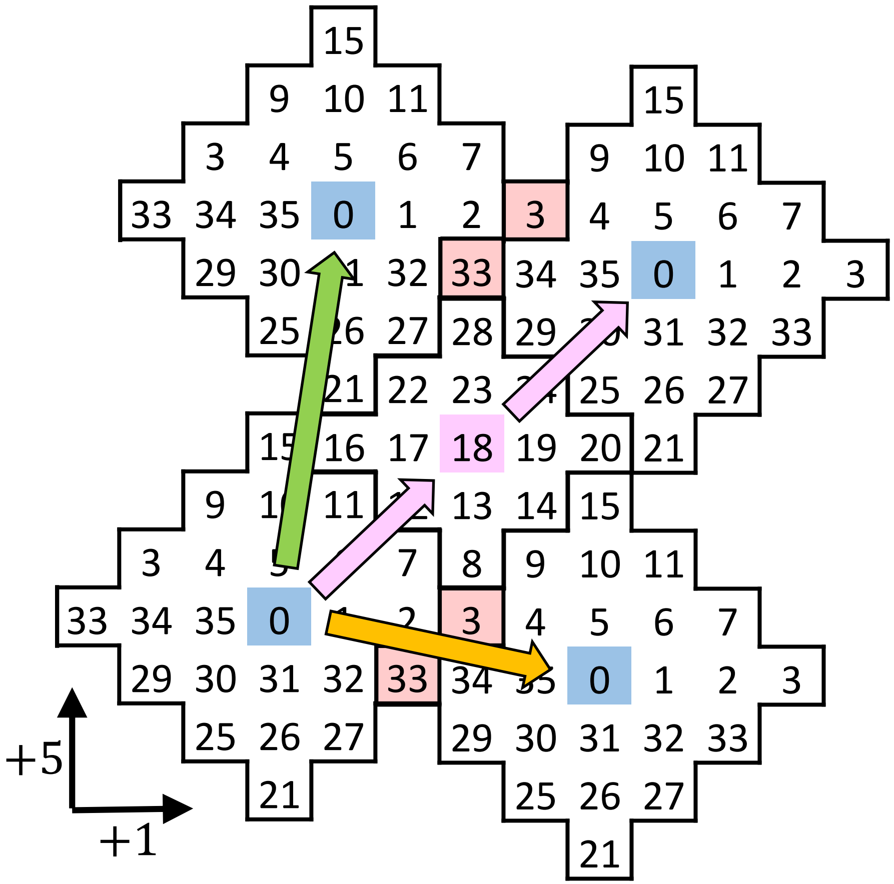
Degree 5, diameter 3, order 36: Minimum-overlap lattice tiling of $\mathbb{Z}^2$ with generator set $\{1, 5\}$



80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101
75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96
70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91
65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86
60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81
55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76
50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71
45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66
40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56
30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

The lattice may also be considered as the sum of a principal lattice and an involutory lattice

The Lee spheres for the extremal degree 5 circulant graph of diameter 3 and order 36



Lattice generator vectors for diameter $k = 3$

➡ $V_1 = (5, -1)$

➡ $V_2 = (1, 7)$

➡ $V_m = (3, 3)$

For any diameter:

$V_1 = (2k - 1, -1)$

$V_2 = (1,\ 2k + 1)$

$V_m = (k,\ k)$

So we have the LGM:

$$M = \begin{pmatrix} 2k-1 & -1 \\ 1 & 2k+1 \end{pmatrix}$$

Area of the lattice unit cell, $|\det M| = 4k^2$, the order of the graph.

The Abelian Cayley Moore bound

The volume of the Lee sphere of dimension f and radius k is an upper bound for the order of an Abelian Cayley graph of degree $d = 2f$ and diameter k.

It is called the even-degree Abelian Cayley Moore bound:

$$M_{AC}(d,k) = \frac{2^f}{f!}k^f + \frac{2^{f-1}}{(f-1)!}k^{f-1} + O\left(k^{f-2}\right).$$

The sum of the volume of the two Lee spheres of dimension f with radius k and radius $k-1$ is an upper bound for the order of an Abelian Cayley graph of degree $d = 2f+1$ and diameter k.

It is called the odd-degree Abelian Cayley Moore bound:

$$M_{AC}(d,k) = \frac{2^{f+1}}{f!}k^f \qquad\qquad + O\left(k^{f-2}\right).$$

These upper bounds are only achieved if the Lee spheres create a perfect tiling of $\mathbb{Z}^f$. In this case, the volume of the lattice unit cell is equal to the volume of the Lee spheres.

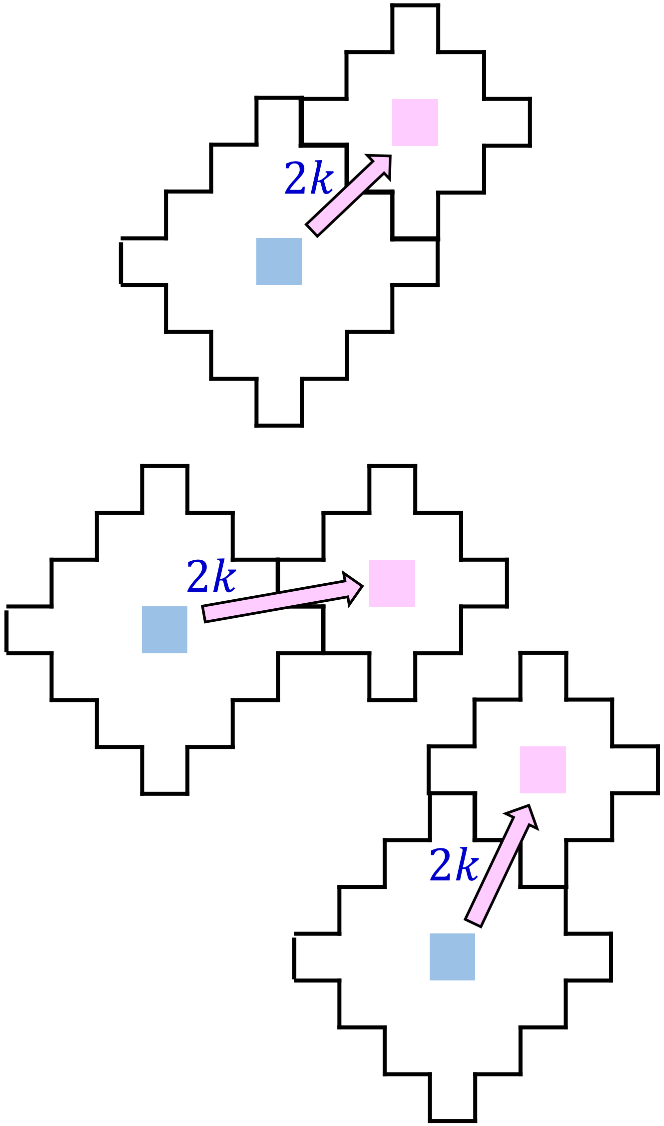
The Golomb-Welch conjecture of 1968 states that such a perfect tiling does not exist for any dimension $f \geq 3$ and radius $k \geq 2$, proved so far for $f = 3, 4, 5$.

The Abelian Cayley Moore bound is achieved by circulant graphs up to degree 4

Degree, d	Moore bound, $M_{AC}(d,k)$	Extremal order, for any diameter k	Defect
2	$2k+1$	$2k+1$	0
3	$4k$	$4k$	0
4	$2k^2+2k+1$	$2k^2+2k+1$	0
5	$4k^2 \quad +2$	$4k^2$	2

According to the Golomb-Welch conjecture for lattice coverings by Lee spheres, the Abelian Cayley Moore bound is not achieved for any degree above 5.

Relative positions of the Lee sphere centres



For a lattice covering, any neighbouring pair of Lee spheres (one of each radius) must either overlap or touch along their common interface.

By the Manhattan metric, the maximum distance between their centres is $2k$:

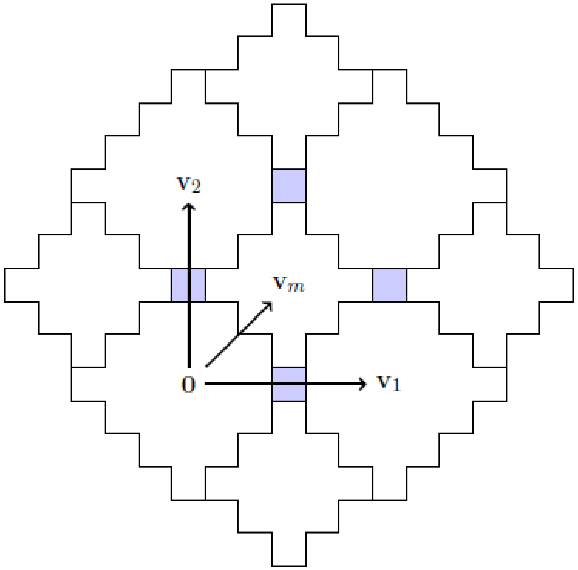
- k within the larger sphere
- $k-1$ within the smaller sphere
- 1 across the common boundary.

So for a maximal lattice covering, we might expect the distance between all such pairs of Lee spheres to be $2k$. In this case, we define the lattice generator matrix to be *radius maximal.*

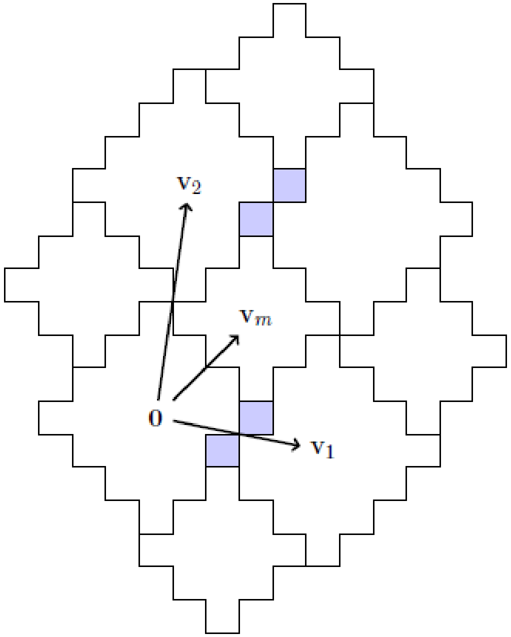
This distance remains constant (Manhattan metric) as the relative positions vary within the same quadrant.

Alternative radius-maximal lattice coverings of $\mathbb{Z}^2$ for degree $d = 5$, diameter $k = 3$

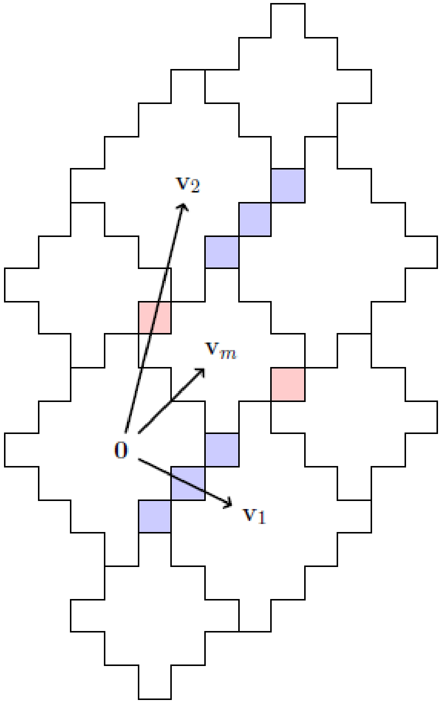
Extremal Abelian Cayley graph of cyclic rank 2



Extremal circulant graph



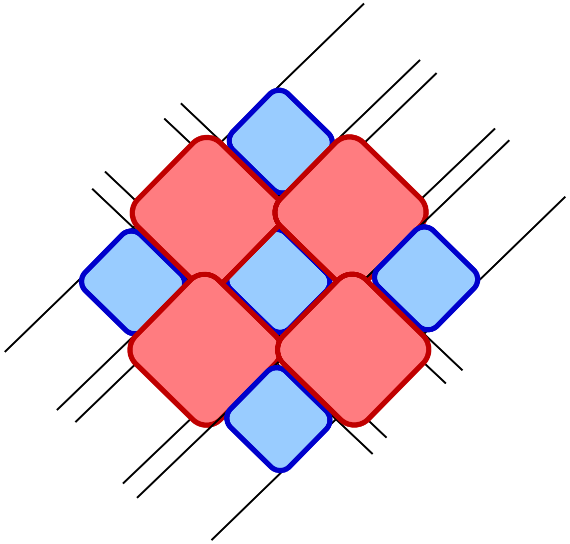
Not a lattice covering of $\mathbb{Z}^2$



LGM:	$\begin{pmatrix} 2k & 0 \\ 0 & 2k \end{pmatrix}$	$\begin{pmatrix} 2k-1 & -1 \\ 1 & 2k+1 \end{pmatrix}$	$\begin{pmatrix} 2k-2 & -2 \\ 2 & 2k+2 \end{pmatrix}$
Determinant:	$4k^2$	$4k^2$	$4k^2$

Key: 🟪 Lee spheres overlap 🟧 No coverage by Lee spheres

Radius-maximal lattices create coverings only if the off-diagonal pairs are $\langle 0, 0\rangle$ or $\langle 1, -1\rangle$



Lee spheres of radius

k: 🟥 $k-1$: 🟦

Extremal Abelian Cayley graph of cyclic rank 2

LGM: $\begin{pmatrix} 2k & 0 \\ 0 & 2k \end{pmatrix}$

Off-diagonal magnitude

4: No covering

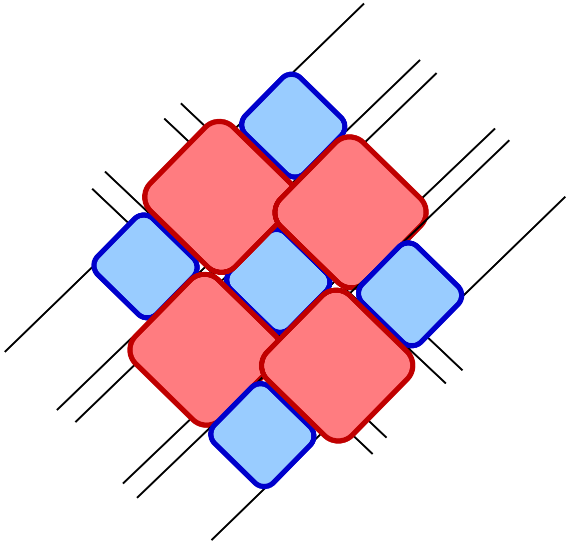
3: No covering

2: No covering

1: Extremal circulant

⇨ 0: Cyclic-rank 2 Abelian Cayley

Radius-maximal lattices create coverings only if the off-diagonal pairs are $\langle 0, 0\rangle$ or $\langle 1, -1\rangle$



Lee spheres of radius

k: 🟥 $k-1$: 🟦

Extremal circulant graph

LGM: $\begin{pmatrix} 2k-1 & -1 \\ 1 & 2k+1 \end{pmatrix}$

Off-diagonal magnitude

4: No covering

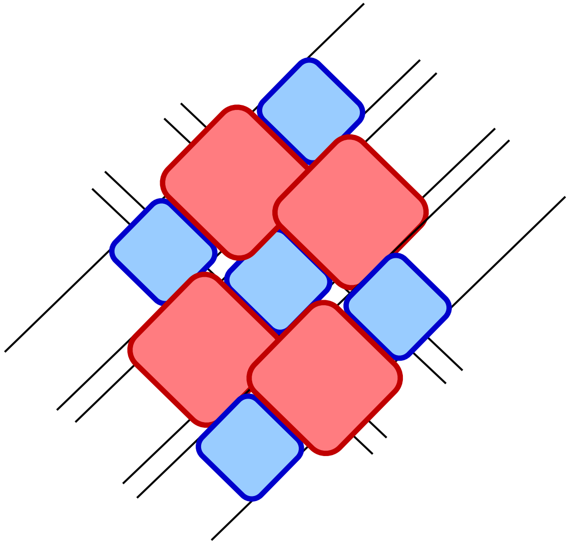
3: No covering

2: No covering

1: Extremal circulant

0: Cyclic-rank 2 Abelian Cayley

Radius-maximal lattices create coverings only if the off-diagonal pairs are $\langle 0,0\rangle$ or $\langle 1,-1\rangle$



Lee spheres of radius

k: 🟥 $k-1$: 🟦

Not a lattice covering of $\mathbb{Z}^2$

LGM: $\begin{pmatrix} 2k-2 & -2 \\ 2 & 2k+2 \end{pmatrix}$

Off-diagonal magnitude

4: No covering

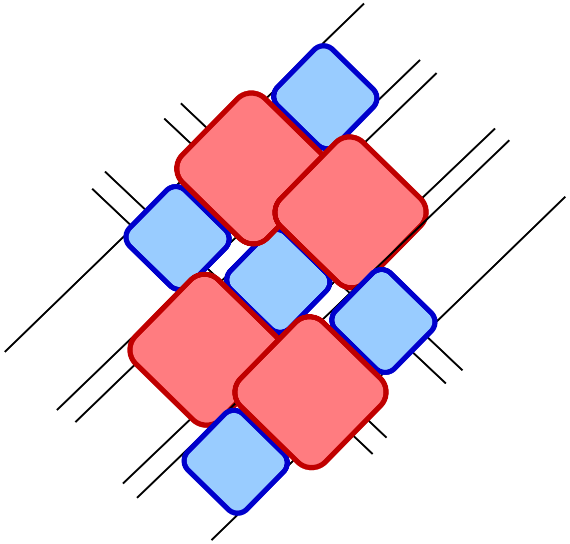
3: No covering

⇨ 2: No covering

1: Extremal circulant

0: Cyclic-rank 2 Abelian Cayley

Radius-maximal lattices create coverings only if the off-diagonal pairs are $\langle 0, 0\rangle$ or $\langle 1, -1\rangle$



Lee spheres of radius

k: 🟥 $k-1$: 🟦

Not a lattice covering of $\mathbb{Z}^2$

LGM: $\begin{pmatrix} 2k-3 & -3 \\ 3 & 2k+3 \end{pmatrix}$

Off-diagonal magnitude

4: No covering

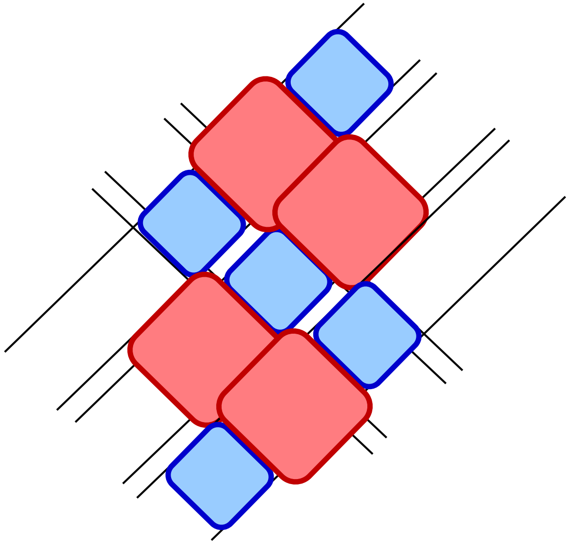
➡ 3: No covering

2: No covering

1: Extremal circulant

0: Cyclic-rank 2 Abelian Cayley

Radius-maximal lattices create coverings only if the off-diagonal pairs are $\langle 0, 0\rangle$ or $\langle 1, -1\rangle$



Lee spheres of radius

k: 🟥 $k-1$: 🟦

Not a lattice covering of $\mathbb{Z}^2$

LGM: $\begin{pmatrix} 2k-4 & -4 \\ 4 & 2k+4 \end{pmatrix}$

Off-diagonal magnitude

➡ 4: No covering

3: No covering

2: No covering

1: Extremal circulant

0: Cyclic-rank 2 Abelian Cayley

Radius-maximal lattices that create coverings of $\mathbb{Z}^2$

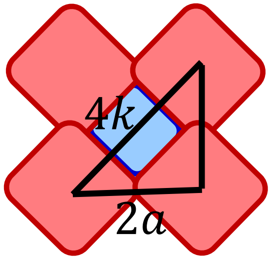
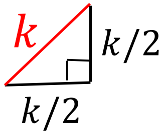
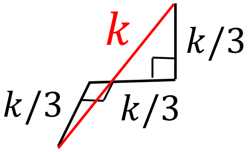
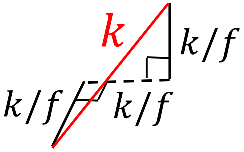
For any radius-maximal lattice that creates a covering of $\mathbb{Z}^2$ for a given diameter k, there is a lattice generator matrix corresponding to a degree 5 Abelian Cayley graph in the following canonical form:

$$\begin{pmatrix} 2k + b_1 & c_{1,2} \\ -c_{1,2} & 2k + b_2 \end{pmatrix}$$

- Each element of the leading diagonal has form $2k + b_i$
- The trace is $4k$, or equivalently $\sum b_i = 0$
- For any off-diagonal element $c_{ij}, i \neq j$,
  $|c_{ij}| = 0$ or 1 and $c_{ij} + c_{ji} = 0$.

Two examples we have seen: $\begin{pmatrix} 2k & 0 \\ 0 & 2k \end{pmatrix}$ $\begin{pmatrix} 2k-1 & -1 \\ 1 & 2k+1 \end{pmatrix}$

Extending the construction to dimension $f > 2$

Dimension	Diagonal length k		Lattice generator matrix (LGM) $(\sum b_i = 0)$	Determinant (quasimaximal order)
2		$2a = 2k$	$\begin{pmatrix} 2k + b_1 & c_{1,2} \\ -c_{1,2} & 2k + b_2 \end{pmatrix}$	$4k^2 + O(1)$
3		$2a = 4k/3$	$\begin{pmatrix} 4k/3 + b_1 & c_{1,2} & c_{1,3} \\ -c_{1,2} & 4k/3 + b_2 & c_{2,3} \\ -c_{1,3} & -c_{2,3} & 4k/3 + b_3 \end{pmatrix}$	$\frac{64}{27}k^3 + O(k)$
f		$2a = 4k/f$	$\begin{pmatrix} 4k/f + b_1 & c_{1,2} & \cdots & c_{1,f} \\ -c_{1,2} & 4k/f + b_2 & \ddots & c_{2,f} \\ \vdots & \ddots & \ddots & \vdots \\ -c_{1,f} & -c_{2,f} & \cdots & 4k/f + b_f \end{pmatrix}$	$\left(\frac{4}{f}\right)^f k^f + O\left(k^{f-2}k\right)$

All extremal and largest-known circulant graph families are quasimaximal

	Odd degree, $d = 2f + 1$	Even degree, $d = 2f$
Quasimaximal order	$\left(\frac{4}{f}\right)^f k^f \ + O\left(k^{f-2}k\right)$	$\frac{1}{2}\left(\frac{4}{f}\right)^f k^f + \left(\frac{4}{f}\right)^{f-1} k^{f-1} + O\left(k^{f-2}k\right)$
Abelian Cayley Moore bound	$\frac{2^{f+1}}{f!}k^f \ + O\left(k^{f-2}\right)$	$\frac{2^f}{f!}k^f \ + \frac{2^{f-1}}{(f-1)!}k^{f-1} \ + O\left(k^{f-2}\right)$
Lattice covering efficiency, R_f	$\frac{2^{f-1}f!}{f^f}$	$\frac{2^{f-1}f!}{f^f}$

$R_1 = 1,\ R_2 = 1,\ R_3 = {}^{8}/_{9},\ R_f \to 0.$

All extremal and largest-known circulant graph families are quasimaximal

Abelian Cayley Moore bound:	Defined by the volume of corresponding Lee spheres, but considered out of context.
Quasimaximal order:	Recognises that for a lattice covering of $\mathbb{Z}^f$ with $f > 2$, the Lee spheres must overlap. For odd degree, there is locally optimal packing of neighbouring pairs of radius k and $k - 1$ Lee spheres. The even-degree case is a translation of odd degree. Extremal circulant graph families are conjectured to be quasimaximal. Conjectures a maximum covering efficiency for any dimension for the Golomb-Welch conjecture.
Lattice generator matrices in canonical quasimaximal format:	Provide the most efficient method known for searching for large circulant graph families and other Abelian Cayley graph families.

One of the first circulant graph families discovered using this approach

Dimension $f = 6$, degree $d = 13$ and diameter class $k \equiv 0 \pmod{3}$.

A largest-known circulant graph family, conjectured extremal, confirmed extremal within the category of quasimaximal families.

Lattice generator matrix (with $a = k/3$):

$$\begin{pmatrix} 2a+1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 2a+1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 2a & -1 & -1 & -1 \\ 0 & 0 & 1 & 2a & 1 & 0 \\ 1 & 1 & 1 & -1 & 2a-1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 2a-1 \end{pmatrix}$$

Order:

$$n = \left(\frac{2k}{3}\right)^6 + 8\left(\frac{2k}{3}\right)^4 + 2\left(\frac{2k}{3}\right)^3 - \left(\frac{2k}{3}\right)^2 - 4\left(\frac{2k}{3}\right)$$

Generating set similarly defined by polynomials in $\frac{2k}{3}$.

About 450,000 LGM trials, compared with 10^{32} gen set trials for diameter 18 graph.

Some discoveries using this approach

Some quasimaximal graph classes

Circulant graph families up to degree 20 for all diameter classes

Abelian Cayley graph families of higher cyclic rank up to degree 15 for all diameter classes

An infinite dynasty (all odd dimensions) of arc-transitive circulant graph families for selected diameters (confirmed up to degree 42)

Some properties of quasimaximal families

Radius-maximal LGM, maximum odd girth $(2k + 1)$

Some relationships between graph families of the same dimension

Transposition, conjugacy, translation, bipartition

Discovery of extremal and largest-known circulant graph families above lower bound