Getting closer to a Moore graph

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Open University Discrete Mathematics Seminar Series

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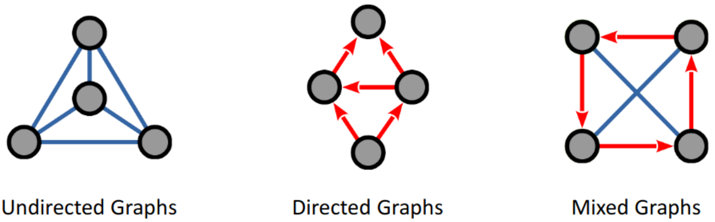
Different graph settings

Interconnection networks have been usually modeled either by graphs or digraphs, depending on the nature of the relationship between nodes in the network. Nevertheless, some networks need the existence at the same time of both, symmetric and nonsymmetric relationships



Degree/Diameter problem for undirected graphs

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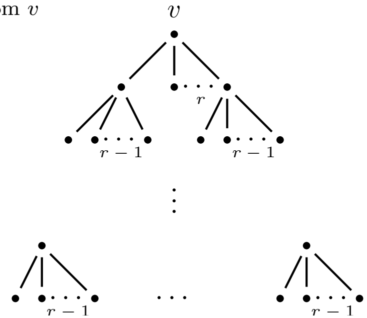

distance from v	number of vertices
0	1
1	r
2	$r(r-1)$
$\vdots$	$\vdots$
$k-1$	$r(r-1)^{k-2}$
k	$r(r-1)^{k-1}$

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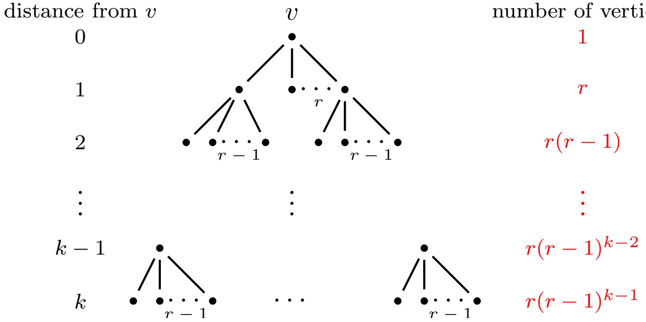

(Moore bound for undirected graphs)

$$n(r,k) \leq M(r,k) = 1 + r + r(r-1) + \cdots + r(r-1)^{k-1}.$$

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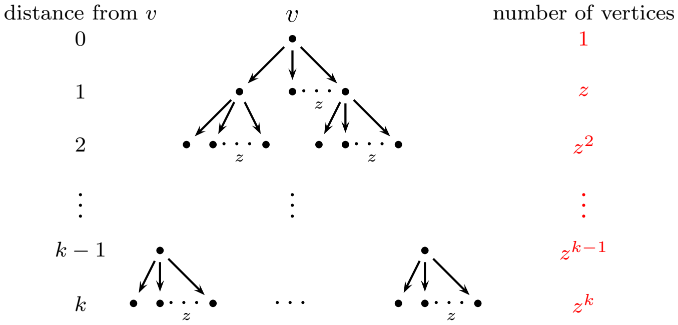

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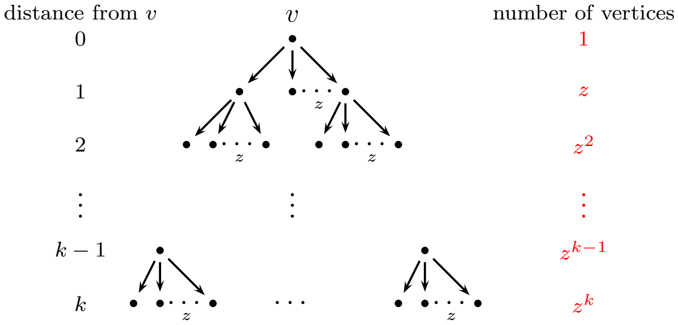

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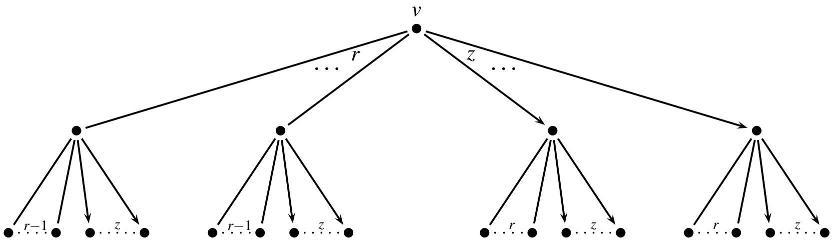

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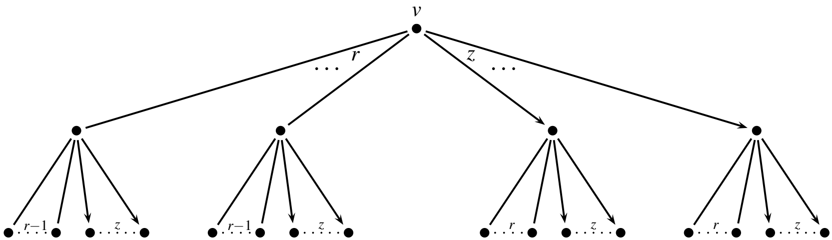

Diameter $k = 2$:
$n(r,z,2) \leq M(r,z,2) = 1 + (r+z) + z(r+z) + r(r+z-1).$

Moore bound for mixed graphs

(Buset, Amiri, Erskine, Miller, Pérez-Rosés (2016); Dalfó, Fiol, L. (2017))

An upper bound for $n(r,z,k)$ can be derived by counting the number of vertices of a particular distance of any given vertex v in a [di]graph with given restrictions. This bound $M(r,z,k)$ is known as the Moore bound.

$$M(r,z,k) = A\frac{u_1^{k+1}-1}{u_1-1} + B\frac{u_2^{k+1}-1}{u_2-1}, \qquad (1)$$

where

$$v = (z+r)^2 + 2(z-r) + 1,$$

$$u_1 = \frac{z+r-1-\sqrt{v}}{2}, \qquad u_2 = \frac{z+r-1+\sqrt{v}}{2},$$

$$A = \frac{\sqrt{v}-(z+r+1)}{2\sqrt{v}}, \qquad B = \frac{\sqrt{v}+(z+r+1)}{2\sqrt{v}}.$$

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- $z = 0$ (no arcs) $\rightarrow$ *Moore graphs* [Banai and Ito '73, Hoffman and Singleton '60, Damerell '73] only exists for $k = 1$ and $r \geq 1$ (Complete graph K_{r+1}) or $k \geq 3$ and $r = 2$ (Cycle graph C_{2k+1}) or
  - $k = 2$ and $r = 2$ (Cycle graph C_5);
  - $k = 2$ and $r = 3$ (Petersen graph);
  - $k = 2$ and $r = 7$ (Hoffman-Singleton graph);
  - $k = 2$ and $r = 57$ (?)

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$r \geq 1$ and $z \geq 1$ $\rightarrow$ *Proper Mixed Moore graphs*

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Theorem (Bosák, 1978)

Let G be a (proper) mixed Moore graph of diameter 2. Then, G is totally regular with directed degree $z \geq 1$ and undirected degree $r \geq 1$. Moreover, there must exist a positive odd integer c such that

$$\textit{Exists odd } c \in \mathbb{Z} \textit{ such that } r = \frac{1}{4}(c^2 + 3) \textit{ and } c | (4z - 3)(4z + 5)$$

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$$I + A + A^2 = J + rI$$

Matrix equation of Mixed Moore graphs of diameter two

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Let $d = r + z$ (total degree). With the mapping $x \to 1 + x + x^2$ and the Moore bound $n = 1 + d + d^2 - r$ we obtain that the factors of $\Phi_A(x)$ are $(x - d)$ and the factors of $(1 + x + x^2 - r)^{\frac{n-1}{2}}$. Two cases to analyze, depending on the irreductibility of $P(x) = (1 + x + x^2 - r)$ in $\mathbb{Q}$:

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- $P(x) = (1 + x + x^2 - r)$ is reducible in $\mathbb{Q}$: Then $4r - 3$ must be a square c^2 in $\mathbb{Z}$. Therefore, the roots of $P(x)$ are $\alpha = \frac{-1+c}{2}$ and $\beta = \frac{-1-c}{2}$. As a consequence,

$$\Phi_A(x) = (x-d)(x-\alpha)^a(x-\beta)^b,$$

where a and b are completely determined by the degrees r and z. Equation $\mathrm{Tr}(A) = 0$ gives the necessary condition given above.

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1. $c = 1 \Rightarrow r = 1$. In this case

$$\Phi(x) = (x-d)x^{d^2-1}(x+1)^d$$

This is the characteristic polynomial of the Kautz digraphs. So a mixed Moore graph exists for any $z \geq 1$. For instance $z = 1$ gives:

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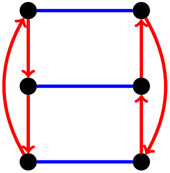

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$$\Phi(x) = (x - 4)(x + 2)^7 (x - 1)^{10}$$

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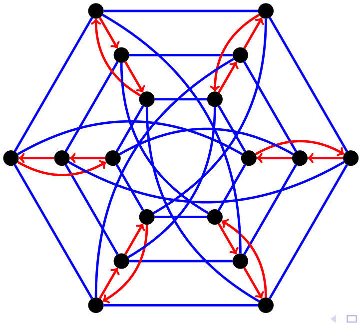

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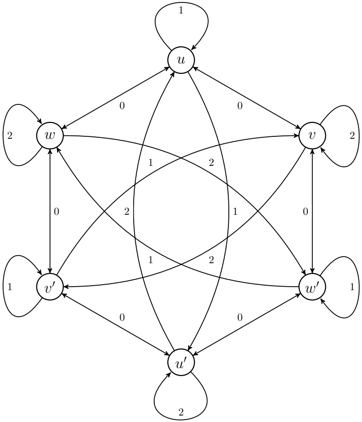
3. Finite geometries: Biaffine planes

Bosák graph constructed by voltage assignament

- $\Gamma = \mathbb{Z}_3$



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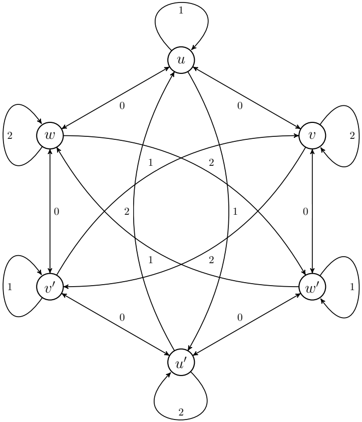
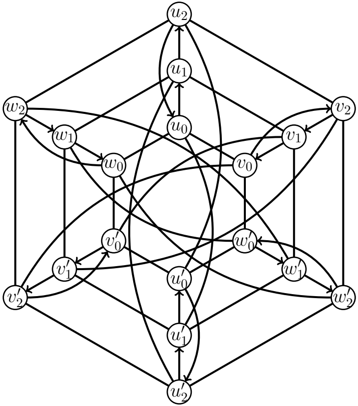


G $\rightarrow$ G^{α}



Mixed Moore graphs of diameter 2. Feasible cases. 1978

[Bosák] Exists odd $c \in \mathbb{Z}$ such that $r = \frac{1}{4}(c^2 + 3)$ and $c | (4z - 3)(4z + 5)$

$M(r, z, 2)$	r	z	d	Existence	Uniqueness
6	1	1	2	Yes	?
12	1	2	3	Yes	?
18	3	1	4	Yes	?
20	1	3	4	Yes	?
30	1	4	5	Yes	?
40	3	3	6	?	?
42	1	5	6	Yes	?
54	3	4	7	?	?
56	1	6	7	Yes	?
72	1	7	8	Yes	?
84	7	2	9	?	?
88	3	6	9	?	?
90	1	8	9	Yes	?
108	3	7	10	?	?
110	1	9	10	Yes	?
⋮	⋮	⋮	⋮	⋮	⋮

Mixed Moore graphs of diameter 2. Feasible cases. 2001

[Gimbert] Enumeration of almost Moore digraphs of diameter 2

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20	1	3	4	Ka$(4,2)$	Yes
30	1	4	5	Ka$(5,2)$	Yes
40	3	3	6	?	?
42	1	5	6	Ka$(6,2)$	Yes
54	3	4	7	?	?
56	1	6	7	Ka$(7,2)$	Yes
72	1	7	8	Ka$(8,2)$	Yes
84	7	2	9	?	?
88	3	6	9	?	?
90	1	8	9	Ka$(9,2)$	Yes
108	3	7	10	?	?
110	1	9	10	Ka$(10,2)$	Yes
⋮	⋮	⋮	⋮	⋮	⋮

Mixed Moore graphs of diameter 2. Feasible cases. 2007

[Nguyen, Miller, Gimbert] On mixed Moore graphs

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Mixed Moore graphs of diameter 2. Feasible cases. 2014

[L., Pérez-Rosés, Pujolàs] mixed Moore Cayley graphs

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⋮	⋮	⋮	⋮	⋮	⋮

Mixed Moore graphs of diameter 2. Feasible cases. 2015

[Jørgensen] New mixed Moore graphs and directed strongly regular graphs

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108	3	7	10	Jørgensen graphs	No
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⋮	⋮	⋮	⋮	⋮	⋮

Mixed Moore graphs of diameter 2. Feasible cases. 2016

[L., Miret, Fern.] Non existence of some mixed Moore graphs using SAT

$M(r,z,2)$	r	z	d	Existence	Uniqueness
6	1	1	2	Ka$(2,2)$	Yes
12	1	2	3	Ka$(3,2)$	Yes
18	3	1	4	Bosák graph	Yes
20	1	3	4	Ka$(4,2)$	Yes
30	1	4	5	Ka$(5,2)$	Yes
40	3	3	6	No	-
42	1	5	6	Ka$(6,2)$	Yes
54	3	4	7	No	-
56	1	6	7	Ka$(7,2)$	Yes
72	1	7	8	Ka$(8,2)$	Yes
84	7	2	9	No	-
88	3	6	9	?	?
90	1	8	9	Ka$(9,2)$	Yes
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Exists odd $c \in \mathbb{Z}$ such that $r = \frac{1}{4}(c^2 + 3)$ and $c|(4z - 3)(4z + 5)$

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132	1	10	11	$\mathrm{Ka}(11,2)$	Yes
150	7	5	12	?	?
154	3	9	12	?	?
156	1	11	12	$\mathrm{Ka}(12,2)$	Yes
180	3	10	13	?	?
182	1	12	13	$\mathrm{Ka}(13,2)$	Yes
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⋮	⋮	⋮	⋮	⋮	⋮

Problem

Prove or dismiss the existence of mixed Moore graphs.

Mixed Moore graphs of diameter $k \geq 3$.

Question (Bosak, 1978)

Are there mixed Moore graphs of diameter $k \geq 3$*?*

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The regularity question has been studied recently by J. Tuite and G. Erskine, showing that every mixed graph of

- diameter $k = 2$ and order $M(r, z, 2) - 1$;
- degrees $r = z = 1$ and order $M(1, 1, k) - 1$

must be regular.

Relaxed Moore graphs

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  - **Radial Moore graphs (of defect δ)**: regular graphs of degree d, order $M(d,k)$, radius k and diameter $k + \delta$.

Degree-relaxed Moore graphs

Example of Near-Moore digraph for parameters $k = 2$ and $z = 2$.

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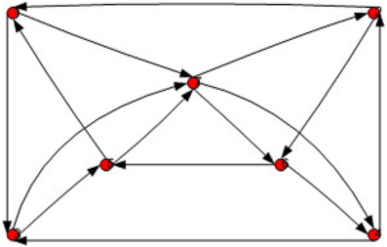

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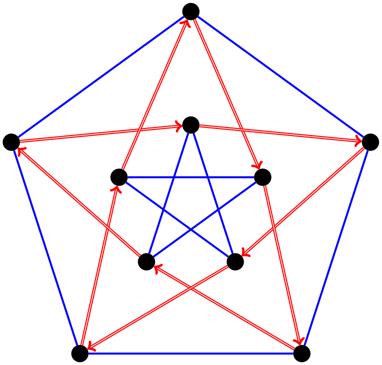

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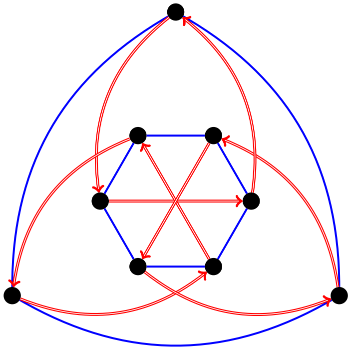

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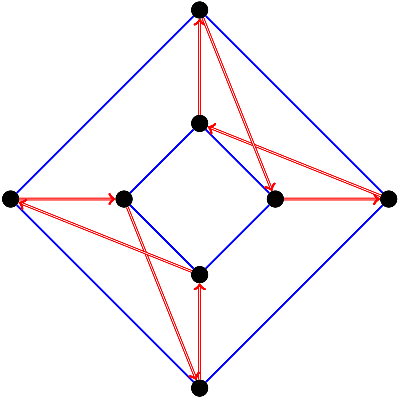

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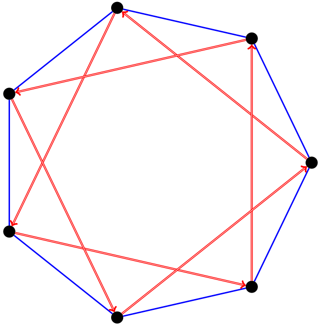
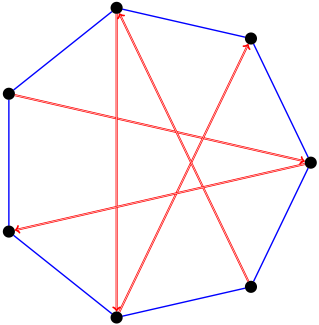


Totally regular



Non regular

Diameter-relaxed Moore graphs

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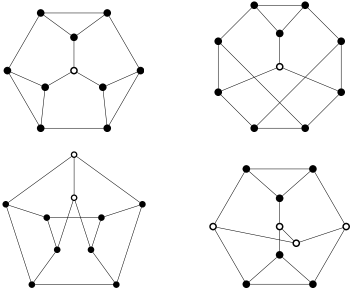

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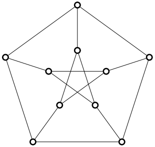

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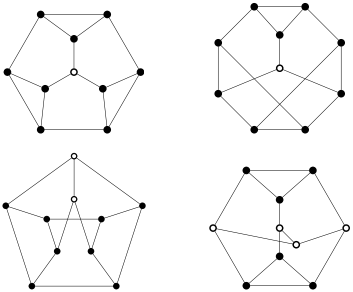

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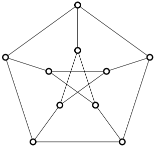

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Enumeration of radial Moore graphs

The enumeration of radial Moore graphs is known for the following cases:

- $k = 2, d = 3$: 5 graphs.
- $k = 3, d = 3$: 1062 graphs.
- $k = 2, d = 4$: 282224 graphs.

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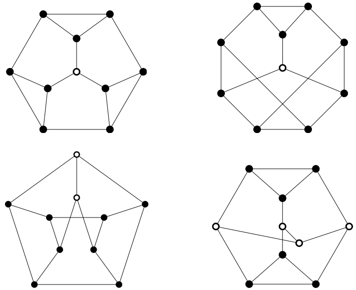
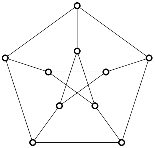


Moore graph for d=3 and k=2



Measuring the closeness to Moore graphs

How to rank all radial Moore graphs according to their ‘proximity’ to being a Moore graph?

Definition

- The *status* of a vertex v is the sum of all distances to a vertex v, $s(v) = \sum_{u \in V} d(u, v)$.
- The *status vector* of G, $\mathbf{s}(G)$, as the vector constituted by the status of all its vertices.

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Given two positive integers d and k, let G be a connected regular graph of degree d and order $n = M(d, k)$. Then, for every vertex v of G we have

$$s(v) \geq \sum_{i=1}^{k} i \cdot d(d-1)^{i-1}$$

This bound is attained for every vertex if and only if G is a Moore graph.

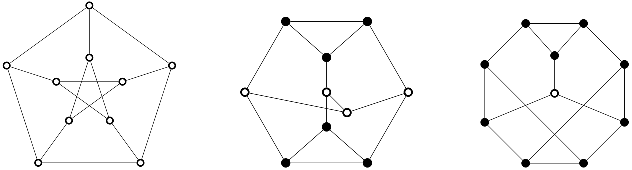
Measuring the closeness to Moore graphs

Let $\mathbf{s}_{d,k}$ be the status vector of a (maybe hypothetical) Moore graph.

Definition

The norm status of a radial Moore graph G of degree d and radius k is

$$N(G) = \|\mathbf{s}(G) - \mathbf{s}_{d,k}\|_1$$



$\mathbf{s}_{3,2} : 15^{10}$

$\mathbf{s}(G_3) : 15^4,\ 17^6$
$N(G_3) = 12$

$\mathbf{s}(G_2) : 15,\ 16^4,\ 17^5$
$N(G_2) = 14$

Since $N(G_3) < N(G_2)$, we say G_3 is closer than G_2 to being a Moore graph.

The closest radial Moore graphs to being a Moore graph

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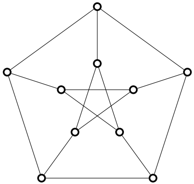
Case: $k = 2, d = 3$ [Complete enumeration gives 5 graphs]



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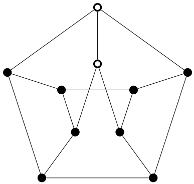

$\mathbf{s}(G_4) : 15^2,\ 16^8$

$N(G_4) = 8$

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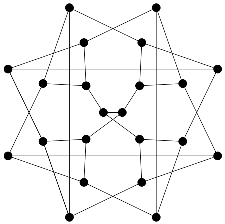
Case: $k = 3, d = 3$ [Complete enumeration gives 1062 graphs]



$\mathbf{s}_{3,3} : 51^{22}$ $\qquad$ $\mathbf{s}(G) : 51^6, 52^8, 53^8$

$N(G) = 24$

The closest radial Moore graphs to being a Moore graph

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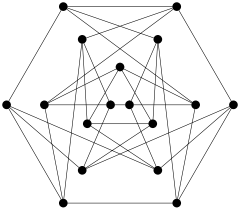
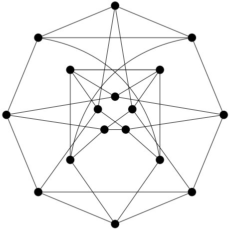
Case: $k = 2, d = 4$ [Complete enumeration gives 282224 graphs]





$\mathbf{s}_{4,2} : 28^{17}$

$\mathbf{s}(\hat{H}_4) : 28^2, 29^{12}, 30^3$
$N(\hat{H}_4) = 18$

$\mathbf{s}(H_4) : 28^8, 30^9$
$N(H_4) = 18$

H_4 belongs to a particular family H_d of radial Moore graphs of radius two where $\mathbf{s}(H_d) = (2d^2 - d)^{2d}, (3d^2 - 6d + 6)^{(d-1)^2}$ and $N(H_d) = (d-1)^2(d-2)(d-3)$.

Radial mixed Moore graphs

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A totally regular mixed graph G of degrees (r,k), radius k, diameter $k+1$ and order the Moore bound is called a radial mixed Moore graph.

Proposition (Ceresuela, L. 2022+)

Given three positive integers $r>1$, $z>1$ and $k>1$, let G be a radial mixed Moore graph of degrees (r,z), order $M(r,z,k)$ and diameter $k+1$. Then, for every vertex v of G we have

$$s(v) \geq A\left(\frac{k\lambda_1^{k+2}-(k+1)\lambda_1^{k+1}+\lambda_1}{(\lambda_1-1)^2}\right)+B\left(\frac{k\lambda_2^{k+2}-(k+1)\lambda_2^{k+1}+\lambda_2}{(\lambda_2-1)^2}\right)$$

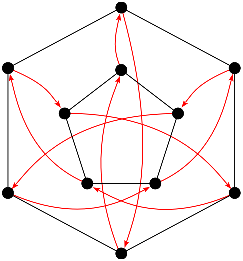
This bound is attained for every vertex if and only if G is a mixed Moore graph.

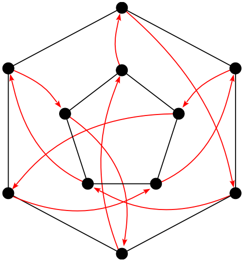
Radial mixed Moore graphs

Radial mixed Moore graphs

Proposition (Ceresuela, L. 2022+)

There are 9486 radial mixed Moore graphs for $(r, z) = (2, 1)$ and radius $k = 2$, being G_1 and G_2 those with minimum status norm.





$\mathbf{s}_{2,1,2} : 17^{11}$

$\text{Spec} = \{3, \lambda_1, \lambda_2\}$

$\mathbf{s}(G_1) : 17^4, 18^6, 19^1$

$N(G_1) = 8$

$\{3, -1, 1, \lambda_1, \lambda_2\}$

$\mathbf{s}(G_2) : 17^4, 18^6, 19^1$

$N(G_2) = 8$

$\{3, -1, 1, \lambda_1, \lambda_2\}$

Radial mixed Moore graphs

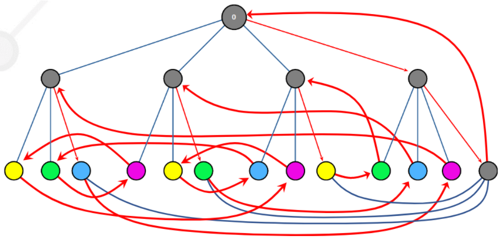
Radial mixed Moore graphs

Proposition (Ceresuela, L. 2022+)

Given any $z \geq 1$, there is a family G_z of $(1, z, 2)$-radial mixed Moore graphs with norm status $N(G_z) = 2$. This is the closest value to being a Moore graph.

Proposition (Ceresuela, L. 2022+)

For every $r > 2$ there exists a family H_r of $(r, 1, 2)$-radial mixed Moore graphs with norm status $N(H_r) = r^4 - 2r^3 + 8r - 2$.



Which graph is the best candidate to become the closest graph to a Moore graph?

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Case $r = 2, z = 1, k = 2$. *The Moore bound is* $M(2, 1, 2) = 11$. *There is no (mixed) Moore graph in this case.*

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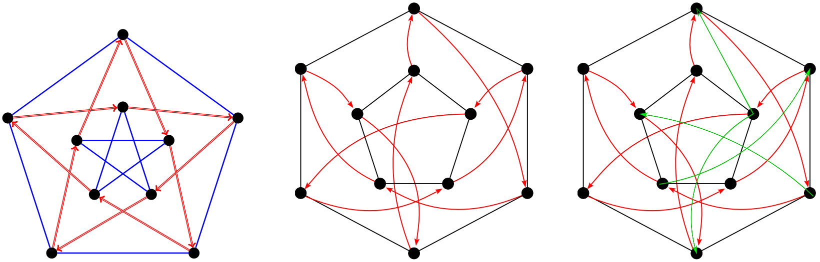

order-relaxed diameter-relaxed degree-relaxed

Some research lines

- Research in some particular versions of the degree/diameter (mixed) problem, namely when the mixed graphs are restricted to a certain class, such as,
  - bipartite mixed graphs.
  - planar mixed graphs.
  - vertex-transitive mixed graphs.
  - mixed Cayley graphs.
  - . . . .
- Define new ranking measures.