Cubic graphs that cannot be covered with four perfect matchings

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based on a joint work with Martin Škoviera

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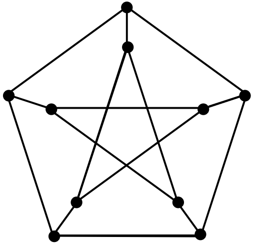
- cubic graphs
  - 3-edge-colourable
  - snarks – cubic graphs that do not admit a 3-edge-colouring



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- snarks are crucial for many conjectures and open problems

Definitions

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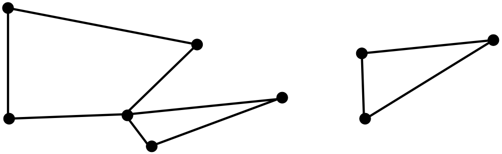

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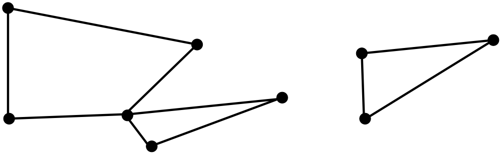
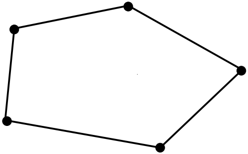


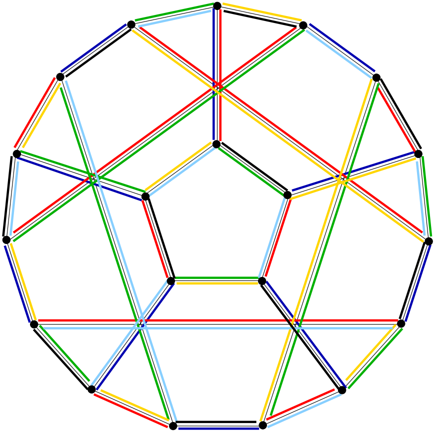
circuit – 2-regular connected graph



5-Cycle Double Dover Conjecture (5CDCC)

Conjecture (Celmins 1984, Preissmann 1981)

Every bridgeless graph admits a collection of five cycles such that each edge belongs to exactly two of them.



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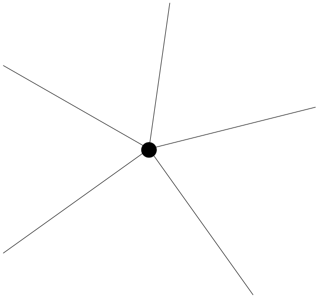

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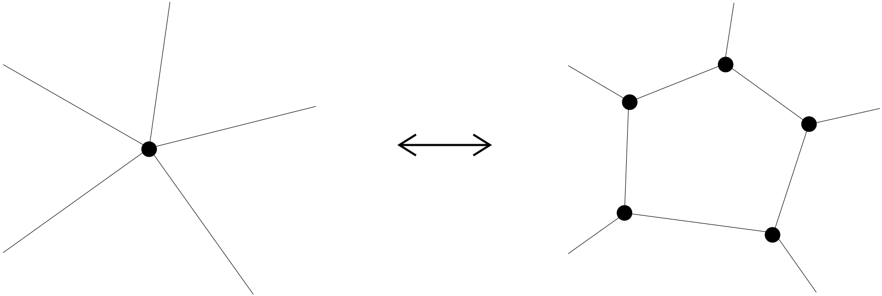

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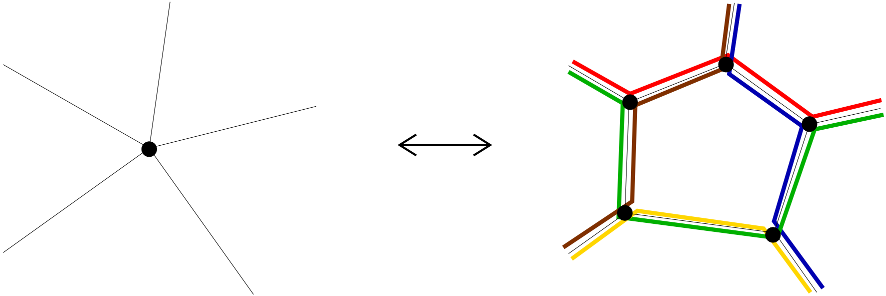

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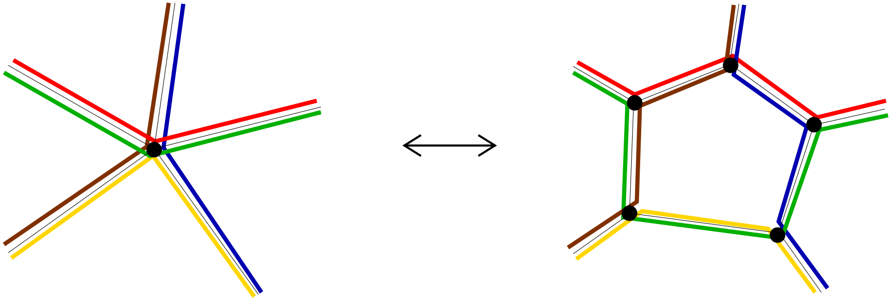

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- crucial role: perfect matchings

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Fulkerson Conjecture (Berge, Fulkerson, 1971)

Every bridgeless cubic graphs contains a family of six perfect matchings that together cover each edge exactly twice.

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  - let M_1, M_2, M_3, M_4 be a PM cover of G
  - E_1– the set of edges covered once
  - $C_i := (M_i \cap E_1) \cup (M_j \cap M_k) \cup (M_j \cap M_l) \cup (M_k \cap M_l)$
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Fan-Raspaud Conjecture, 1994

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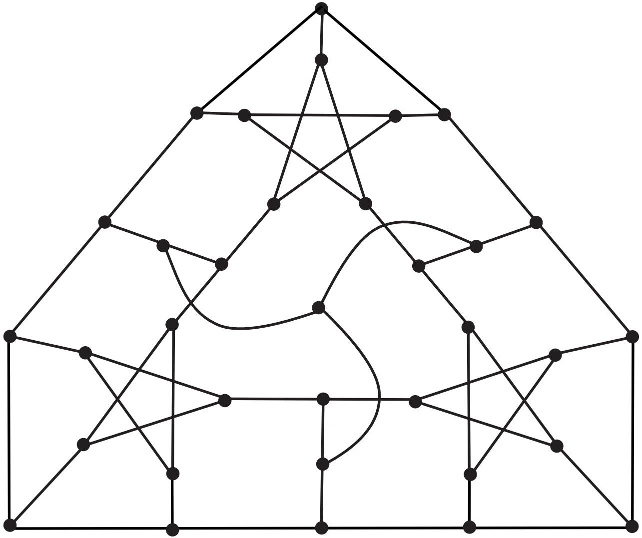
A snark of order 34 with $\pi(G) = 5$



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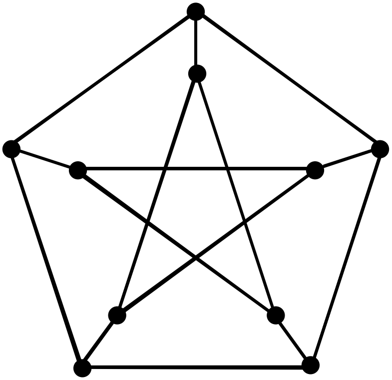

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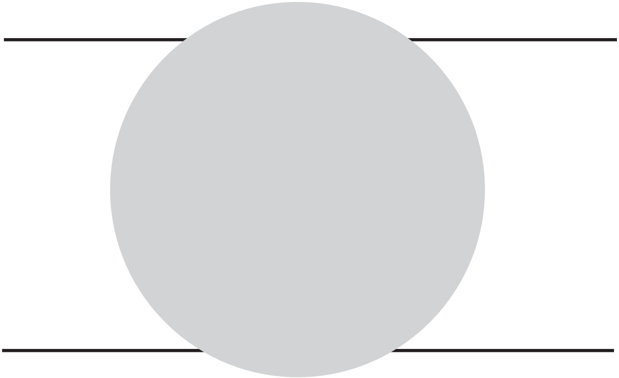

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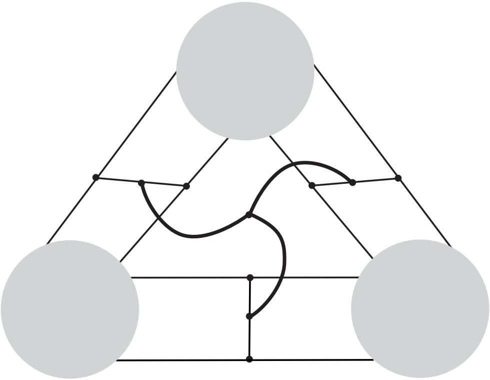

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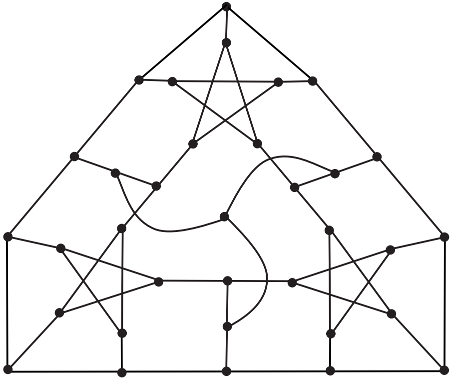

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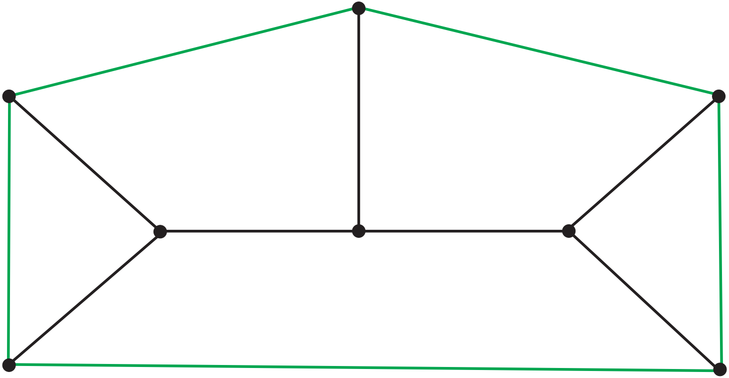

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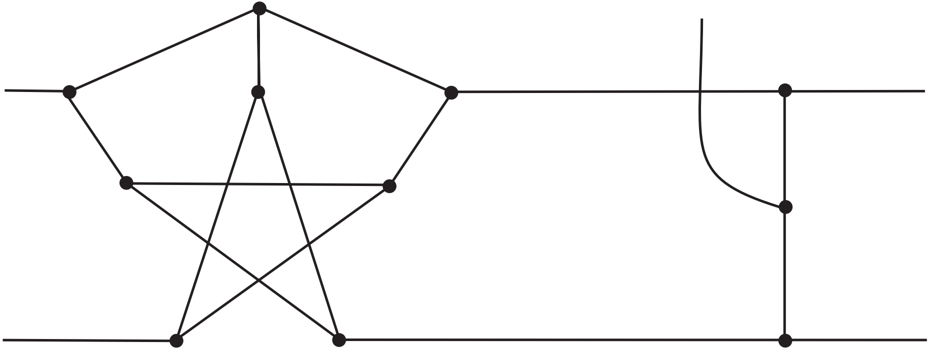

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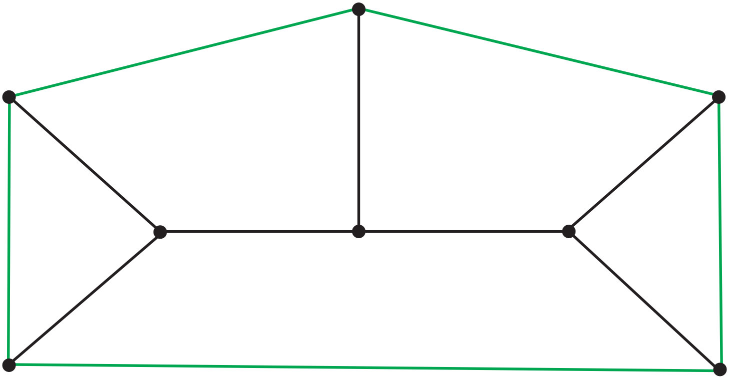

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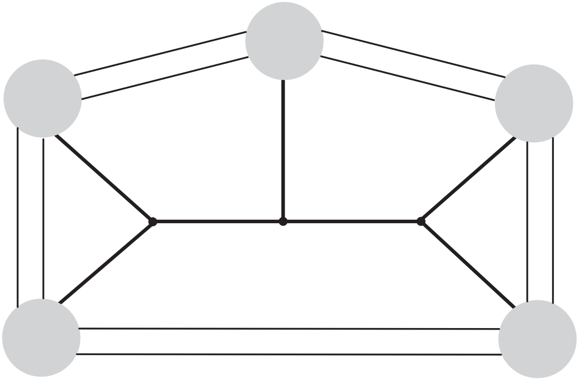

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- Proofs heavily depend on computer-aided arguments.

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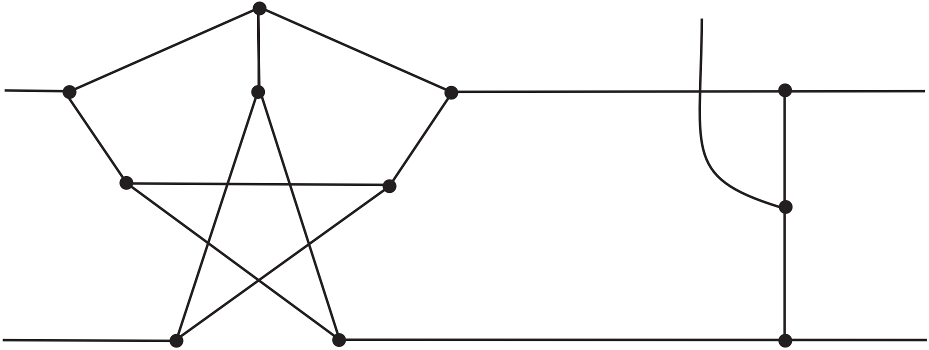

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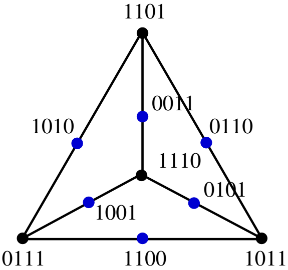
8.1 The pattern set of the Petersen fragment

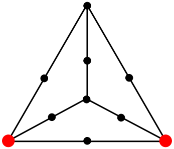
The pattern set of F_0 (42 patterns):

A A AB AC AD	A B CD AB AB	A BC D BC BC
A A AB C D	A B CD AC AC	A BC D BD BD
A AB A AC AD	A B CD C C	A BC D D D
A AB A BC BD	A B CD CD CD	AB AB AB AC AD
A AB AC A AD	A BC A AB BD	AB AC AB AB AD
A AB AC B BD	A BC B AB AD	AB AC AB BC CD
A AB AC C CD	A BC B BC CD	AB AC AD A A
A B AB AB CD	A BC BD A AB	AB AC AD AB AB
A B AB AC BD	A BC BD BC C	AB AC AD AD AD
A B AC A D	A BC BD BD D	AB AC AD B B
A B AC AB BD	A BC D A A	AB AC AD BC BC
A B C A AD	A BC D AB AB	AB AC AD BD BD
A B C C CD	A BC D AD AD	AB AC AD D D
A B CD A A	A BC D B B	AB CD AC AB BC

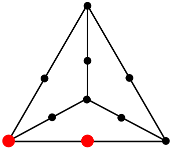
Tetrahedral $\mathbb{Z}_2^4$-flow



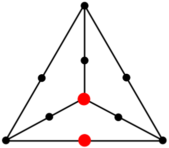
Types of connectors



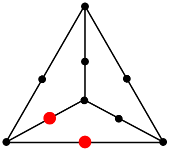
line segment



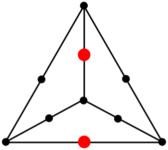
half–line



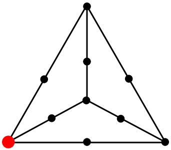
altitude



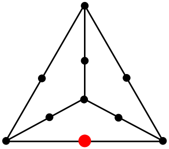
angle



axis



double point



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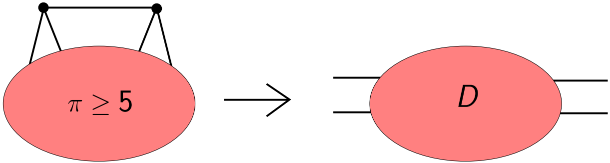
Special types of multipoles



Let D be a $(2,2)$-pole created by the removal of two adjacent vertices from a snark G with $\pi(G) \geq 5$ and by grouping the edges formerly incident with the same vertex to the same connector. Then

$$\mathbf{T}(D) \subseteq$$

$$\{\texttt{dpt} \to \texttt{dpt}, \texttt{alt} \to \texttt{alt}, \texttt{ax} \to \texttt{ax}, \texttt{ang} \to \texttt{ang}, \texttt{ang} \to \texttt{ls}, \texttt{ls} \to \texttt{ang}\}$$

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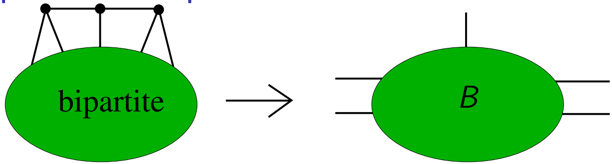
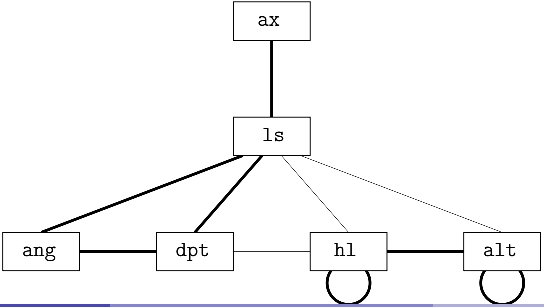
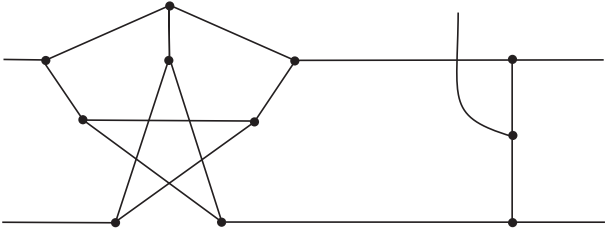
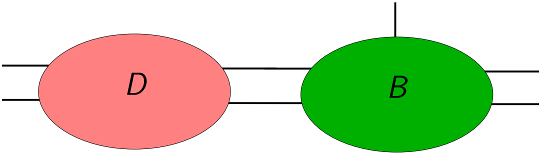
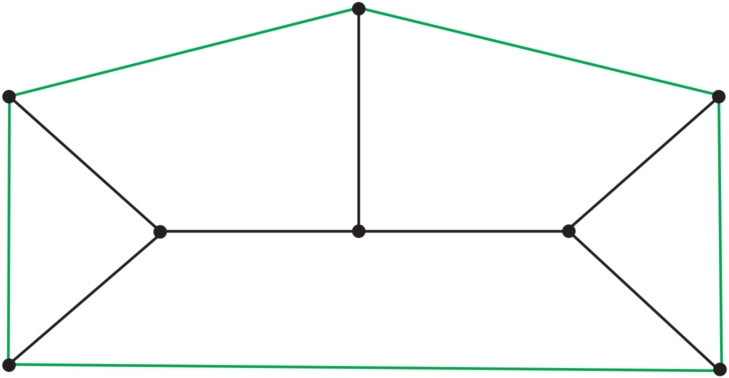


Let B be a $(2,2;1)$-pole created by the removal a path uwv and by grouping the edges formerly incident with u to one connector, the edges formerly incident with v to another connector. Then $\mathbf{T}(B)$ is a subset of transitions depicted in the Figure.

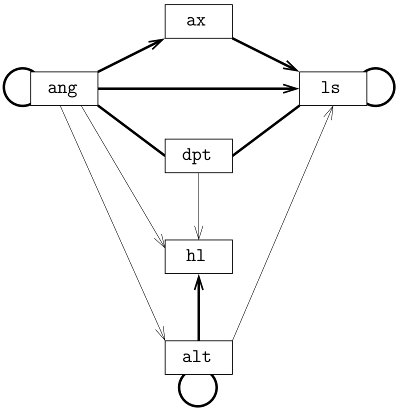





D
B



Induction



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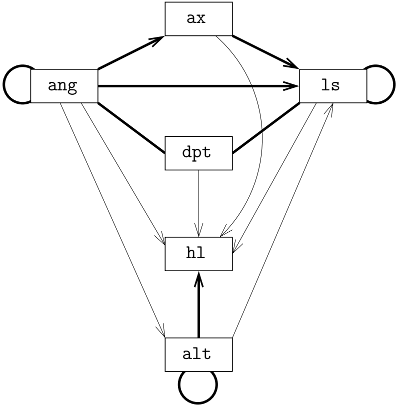

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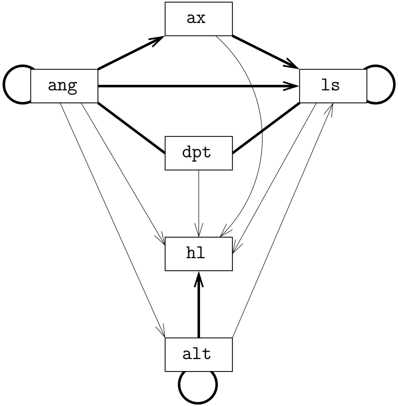

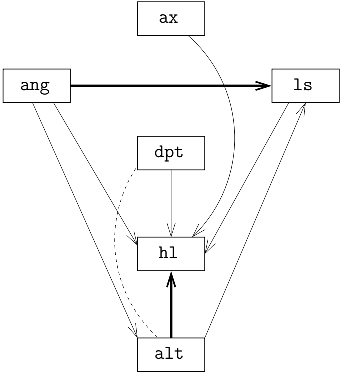

π and cfn

Circular flow number

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Tutte's 5-flow conjecture, 1954

Every bridgeless graphs admits a nowhere-zero (circular) 5-flow.

Cubic graphs with $\pi \geq 5$

- infinite families:
  - windmill graphs [Esperet, Mazzuoccolo; 2014]
  - treelike snarks [Abreu, Kaiser, Labbate, Mazzuoccolo, 2016]
  - a family of Chen [Chen; 2016]
  - Halin snarks [EM, Škoviera; 2021]

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we give a negative answer to this question

Construction

Let G be a bipartite cubic graph. A new graph $\tilde{G}$ is constructed as follows.

- replace each vertex of G with a pair of vertices
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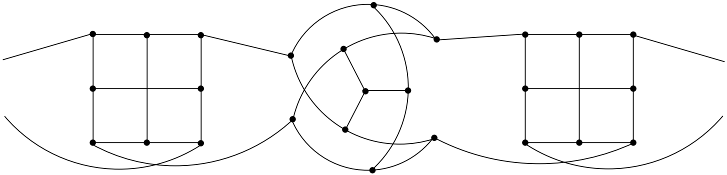

The result

Theorem (EM, Škoviera; 2021)

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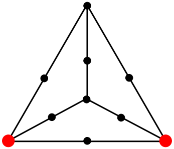
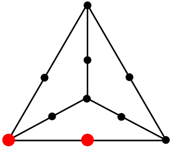
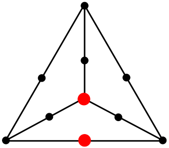


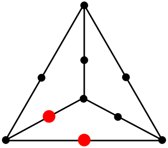
line segment



half–line



altitude



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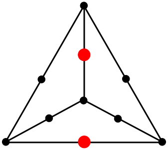
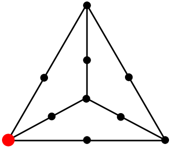
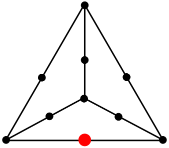


axis



double point



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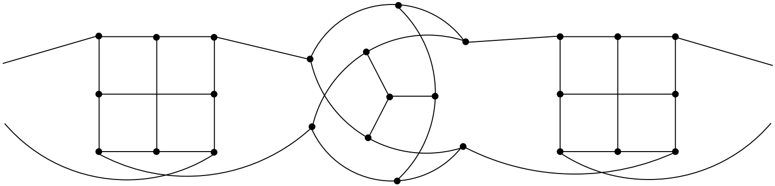
Proof, Part 1: $\pi(\tilde{G}) \geq 5$



- $\mathbf{T}(D_i) \subseteq \{\texttt{dpt} \to \texttt{dpt}, \texttt{alt} \to \texttt{alt}, \texttt{ax} \to \texttt{ax}, \texttt{ang} \to \texttt{ang}, \texttt{ang} \longleftrightarrow \texttt{ls}\}$
- $\mathbf{T}(Q) \subseteq \{\texttt{hl} \to \texttt{hl}, \texttt{ls} \to \texttt{ls}, \texttt{alt} \to \texttt{alt}, \texttt{ang} \longleftrightarrow \texttt{ls}\}$
- we conclude that $\mathbf{T}(D_1 \circ Q \circ D_2) \subseteq \{\texttt{ls} \longleftrightarrow \texttt{ang}, \texttt{ang} \to \texttt{ang}, \texttt{alt} \to \texttt{alt}\}$
- each transition has at least two heavy dangling edges (covered with two PMs)

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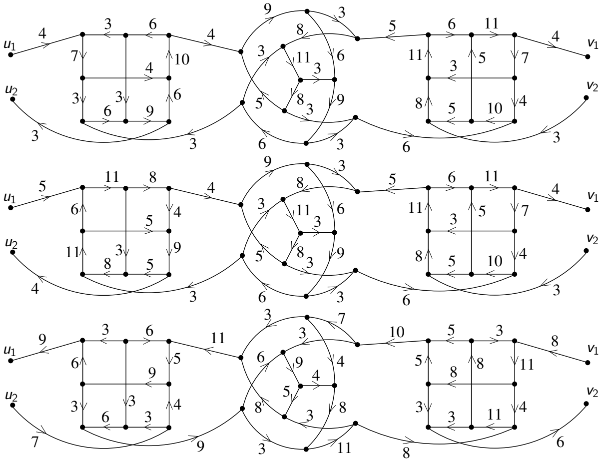
- the set W can be decomposed into m sets according to which superedge it belongs
- average number of heavy edges for a superedge is

$$\frac{2n}{m} = \frac{4}{3} < 2,$$

a contradiction

Proof, Part 2: $cfn(\tilde{G}) < 5$



- $cfn(\tilde{G}) \leq 4 + \frac{2}{3}$

Problem

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What is the infimum of the set of all real numbers r such that there exists a cubic graph G with $\pi(G) \geq 5$ and $\Phi_c(G) = r$?

Thank you for your attention!