Ramsey Minimal Graphs for Pairs Involving a Cycle and a Star

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This talk is based on joint work with Hilda Assiyatun and Edy Tri Baskoro

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08 January 2025

Outline

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Introduction

The Notation

For any simple graphs F, G, and H,

• The notation $F \to (G, H)$ means that for any red-blue coloring on the edges of F, there is a red copy of G or a blue copy of H in F.

• The notation $F - e \nrightarrow (G, H)$ means that there is a red-blue coloring on the edges of $F - e$, so that there is neither a red copy of G nor a blue copy of H in $F - e$.

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Definition of Ramsey minimal graph

If a graph F satisfies that $F \rightarrow (G, H)$ and $F - e \nrightarrow (G, H)$ for any $e \in F$, then F is called a *Ramsey (G, H)-minimal graph.*
The set of all Ramsey (G, H)-minimal graphs is denoted by $\mathcal{R}(G, H)$.

Example of RMG

♣ Example 1
$F_1 \in \mathcal{R}(P_3, P_6)$

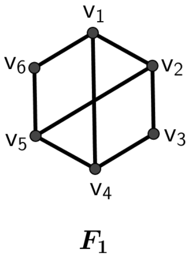
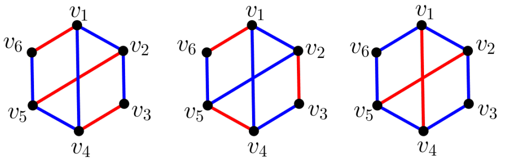


Figure 1: Graph F_1.

Proof.

- Consider any red-blue coloring on the edges of F_1 without a red copy of P_3.
- Then, all red edges will form a matching
- So that, we obtain a blue copy of P_6.



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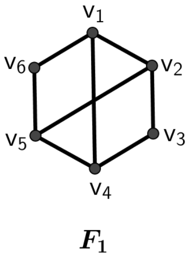

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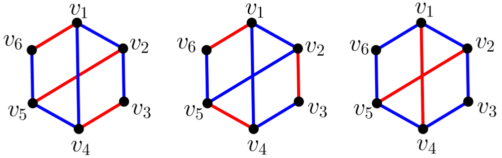
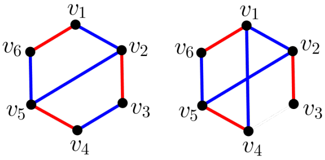


• The minimality:
$\forall e \in E(F_1),\ F_1^* := F_1 - e \nrightarrow (P_3, P_6)$.



Previous Results

- A pair of (G, H) is called *Ramsey-finite* if the set $\mathcal{R}(G, H)$ is finite.
- Otherwise, the pair (G, H) is called *Ramsey-infinite*.

Authors	Year	Results
Baskoro et al	2008	Infinite class of $\mathcal{R}(P_3, C_4)$
Tomas et al	2010	Ramsey $(K_{1,2}, C_4)$-minimal graphs with diameter ≥ 4
Baskoro and Wijaya	2016	Some disconnected graphs in $\mathcal{R}(mK_2, H)$ If $F \in \mathcal{R}(mK_2, P_3)$, then $\mathcal{R}((m+1)K_2, P_3)$
Wijaya and Baskoro	2016	Construct graphs in $\mathcal{R}(2K_2, 2H)$ by using graph theoretical operations over graphs in $\mathcal{R}(2K_2, 2H)$, if H is either a cycle, a path, or a star.
Nabila and Baskoro	2021	$\mathcal{R}(C_n, K_{1,2})$; $n = 5, 6, 8$ Ramsey $(C_n, K_{1,2})$-graphs for even $n \in [10, 18]$.
Hadiputra and Silaban	2021	an infinite family of graphs belongs to $\mathcal{R}(K_{1,2}, C_4)$
Baskoro et al	2022	The characterization of a unicyclic Ramsey (mK_2, P_4)-minimal graphs
Nabila et al	2022	A finite and an infinite classes of graphs which belong to $\mathcal{R}(C_4, K_{1,n})$
Assiyatun et al	2023	Ramsey $(C_4, K_{1,n})$-minimal graphs that constructed by modifying a weighted tree graph

Table 1: Some previous results of Ramsey (G, H)-minimal

Definition of theta-unicyclic

Let U be an edge-weighted graph U on m edges $e_1, e_2, \ldots, e_m$ with the edge weights $a_1, a_2, \ldots, a_m$, respectively. Define a theta-unicyclic graph $\theta[U]$ as follows.

Definition 1

The *theta-unicylic graph* based on U, denoted by $\theta[U]$, is the graph constructed from U by replacing each edge e_i by a union of a_i internally disjoint paths of length 2 for each $i \in [1, m]$.

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From Definition 1, we have $V(\theta[U]) = V(U) \cup \bigcup_{i=1}^{m} A_i$,
where $A_i = \{u_{i,j} |\ i \in [1, m], j \in [1, a_i]\}$.

- The vertices in A_i are called the *internal vertices* of the $\theta[U]$ graph $\forall i \in [1, m]$.
- The *sum* of a unicyclic graph U, denoted by $\mathrm{sum}(U)$, is defined as the sum of all edge weights, namely $\mathrm{sum}(U) = \sum_{i=1}^{m} a_i$.

Example of theta-unicylic graph

Figure 2 illustrates the theta-unicyclic graph $\theta[U]$ obtained from an edge-weighted unicyclic graph U on 10 edges.

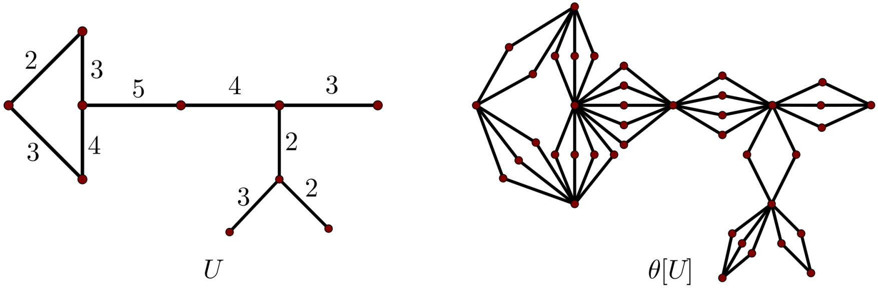

Figure 2: An edge-weighted unicyclic graph U on 10 edges and the corresponding graph $\theta[U]$.

The sufficient conditions

Theorem 1

Let n, m be natural numbers and U be an edge-weighted unicyclic graph on m edges $e_1, e_2, \ldots, e_m$ with the edge weights $a_1, a_2, \ldots, a_m$, respectively, where $m \geq 3$, $n \geq 2$, and $2 \leq a_i \leq 2n - 1$. If the following statements hold

(a) $\text{sum}(U) = mn + 1$,

(b) $\text{sum}(U') < ln + 1$ *for each proper unicyclic subgraph U' of U induced by any l edges,*

(c) $\text{sum}(U') < (l + 1)n$ *for each proper subgraph U' of U induced by any l edges which is a tree,*

then $\theta[U]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph.

The sufficient conditions

Proof of Theorem 1: $\theta[U] \to (C_4, K_{1,n})$

- Let U be an edge-weighted unicyclic graph on m edges with $\text{sum}(U) = mn + 1$.
- Consider any red-blue coloring α on the edges of $\theta[U]$ without blue star $K_{1,n}$.

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- Let B be the set of blue edges under coloring α on $\theta[U]$.
- Since there is no blue $K_{1,n}$, then

$$|B| \leq m(n-1) = mn + 1 - (m+1) = \mathrm{sum(U) - (m+1)} \qquad (1)$$

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$$|B| \leq m(n-1) = mn + 1 - (m+1) = \text{sum(U)} - (\text{m} + 1) \quad (1)$$

- It means that there are at least $(m+1)$ internal vertices in $\theta[U]$ that are not incident to any blue edge.
- So, there are at least two internal vertices in A_j for some $j \in [1, m]$ that only incident to red edges and these vertices will be used to construct a red C_4.
- Therefore, $\theta[U] \to (C_4, K_{1,n})$.

Proof of Theorem 1: $\theta[U] - e \nrightarrow (C_4, K_{1,n}), \forall e \in \theta[U]$

- Let $e = uv$, where u is an internal vertex in A_j for some j and v is a non-internal vertex.
- Define a red-blue coloring on the edges of $\theta[U] - e$ such that u is incident with a red edge and the rest edges are colored by satisfying these conditions

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(i) there is exactly one vertex in A_i that only incident with red edges and each remaining internal vertex in A_i is incident with exactly one blue edge, such that the number of blue edges incident with vertices in A_i is exactly $a_i - 1$ for each $i \neq j$,

(ii) there are exactly two vertices in A_j that are incident only with red edges, one of them is u and each remaining internal vertex in A_j is incident with exactly one blue edge, such that the number of blue edges incident with vertices in A_j is exactly $a_j - 2$, and

(iii) the number of blue edges incident to each non-internal vertex is exactly $n - 1$.

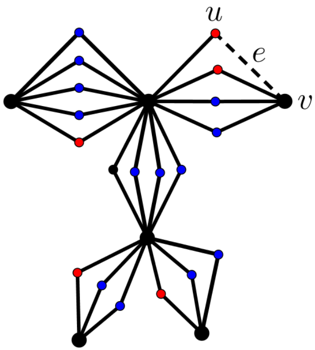

Figure 3: Example the red-blue coloring process.

Proof of Theorem 1: $\theta[U] - e \nrightarrow (C_4, K_{1,n}), \forall e$

- Let B be the set of all blue edges in $\theta[U] - e$ by the above coloring.
- From (i), (ii) we obtain $|B| = (a_j - 2) + \sum_{i=1 \text{ and } i \neq j}^{m} (a_i - 1)$ and from (iii) we obtain $|B| = m(n-1)$.
- Remember that each blue edge must be incident to one internal vertex and one non-internal vertex so that we have

$$(a_j - 2) + \sum_{i=1 \text{ and } i \neq j}^{m} (a_i - 1) = |B| = m(n-1). \tag{2}$$

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- It means that the above red-blue coloring can always be constructed from conditions (i), (ii), and (iii).
- Condition (iii) shows that there is no blue $K_{1,n}$.
  Conditions (i) and (ii) show that we only have exactly one internal vertex incident to two red edges. Therefore, there is no red C_4 in $\theta[U] - e$.
- Thus, $\theta[U] - e \nrightarrow (C_4, K_{1,n})$ for any edge e in $\theta[U]$.

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- Thus, $\theta[U] - e \nrightarrow (C_4, K_{1,n})$ for any edge e in $\theta[U]$.
- Note: Condition (b) is required, since the existence of a unicyclic subgraph U' of size l satisfying $\text{sum}(U') = ln + 1$ would imply $\theta[U'] \rightarrow (C_4, K_{1,n})$. Similarly, Condition (c) is also required.

Main Results

Theorem 2

Let U be an edge-weighted unicyclic graph on m edges $e_1, e_2, \ldots, e_m$ with the edge weights $a_1, a_2, \ldots, a_m$, respectively and $\mathrm{sum}(U) = mn + 1$. Then, the graph $\theta[U]$ is not a Ramsey $(C_4, K_{1,n})$-minimal graph if

(i) there is an edge e_i with $a_i \geq 2n$,

(ii) there are two adjacent edges e_i and e_j with $a_i + a_j \geq 3n$,

(iii) there are three edges e_i, e_j, and e_k form a cycle C_3 with $a_i + a_j + a_k \geq 3n + 1$, or

(iv) there is a pendant edge e_i with $a_i \leq n$.

Some edge-weight sequences for theta-unicyclic graphs

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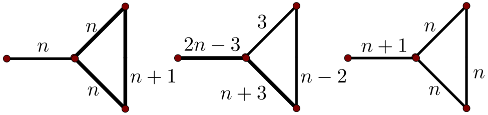

U_1 U_2 U_3

Figure 4: Some distribution of the edge weights in a unicyclic graph on 4 edges.

- From Theorem 1, we know that for a unicyclic graph U on 4 edges, $\text{sum}(U) = 4n + 1$. It is easy to verify that $\text{sum}(U_1) = \text{sum}(U_2) = \text{sum}(U_3) = 4n + 1$.

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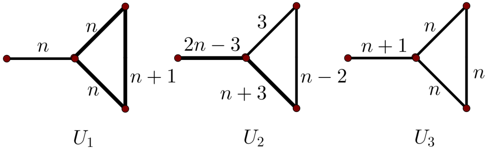

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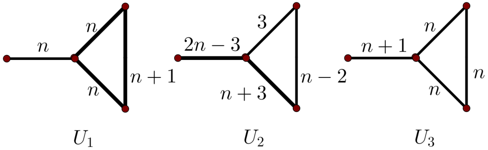

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- Graph U_1 does not satisfy the second condition, since there is a proper unicyclic subgraph $U_1' \cong C_3$ with $\text{sum}(U_1') = 3n + 1$.
- Graph U_2 does not satisfy the third condition, since there is a proper subgraph $U_2' \cong P_3$ with $\text{sum}(U_2') = 3n$.

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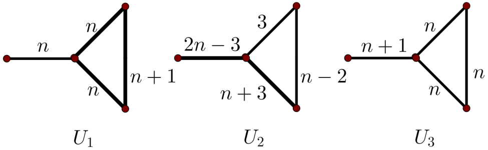

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- Graph U_2 does not satisfy the third condition, since there is a proper subgraph $U_2' \cong P_3$ with $\text{sum}(U_2') = 3n$.
- Graph U_3 satisfies the second and third conditions.
- Thus, $\theta[U_1], \theta[U_2]$ are not Ramsey $(C_4, K_{1,n})$-minimal graphs and $\theta[U_3]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph.

Some classes of theta-unicyclic graphs which belong to $\mathcal{R}(C_4, K_{1,n})$

Corollary 1

Let U be an edge-weighted unicyclic graph on m edges $e_1, e_2, \ldots, e_m$ with the edge weights $a_1, a_2, \ldots, a_m$ and let e_1 be an edge on the cycle. The graph $\theta[U]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph, if the weights $a_1 = n - m + 2$ and $a_i = n + 1$ for $i \in [2, m]$, where $3 \leq m \leq n$.

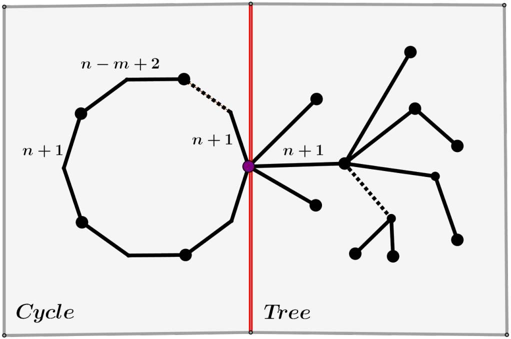

Figure 5: Illustration for Corollary 1.

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Corollary 2

Let U be an edge-weighted unicyclic graph on m edges $e_1, e_2, \ldots, e_m$ with the edge weights $a_1, a_2, \ldots, a_m$ and let e_1, e_2 be two edges on the cycle. The graph $\theta[U]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph, if the weights $a_1 = 2 + k$, $a_2 = 2n - m + 1 - k$, and $a_i = n + 1$ for $i \in [3, m]$, where $3 \leq m \leq n$ and $0 \leq k \leq \left\lfloor \frac{n-2}{2} \right\rfloor$.

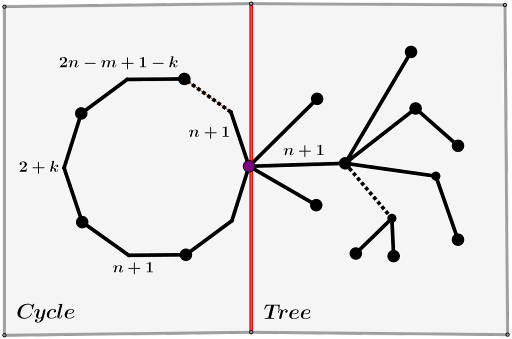

Figure 6: Illustration for Corollary 2.

Some classes of theta-unicyclic graphs which belong to $\mathcal{R}(C_4, K_{1,n})$

Corollary 3

Let U be an edge-weighted unicyclic graph on m edges $e_1, e_2, \ldots, e_m$ with the edge weights $a_1, a_2, \ldots, a_m$ and let e_1, e_2, e_3 be three edges on the cycle. The graph $\theta[U]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph, if the weights $a_1 = 1 + k_1$, $a_2 = n + 2 + k_2$, $a_3 = 2n - m + 1 - k$, and $a_i = n + 1$ for $i \in [4, m]$, where $n \geq 6$, $3 \leq m \leq n - k - 2$, $1 \leq k_1 \leq k \leq \lfloor \frac{n-2}{2} \rfloor$, $0 \leq k_2 \leq n - m - k - 3$, $k_1 > k_2$, and $k = k_1 + k_2$.

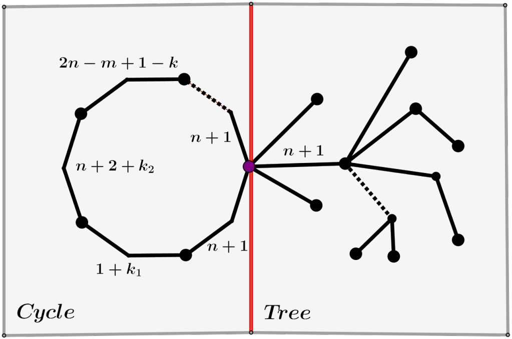

Figure 7: Illustration for Corollary 3.

Some classes of theta-unicyclic graphs which belong to $\mathcal{R}(C_4, K_{1,n})$

Besides those above edge-weight sequences, we give another edge-weight sequence for cycles which is shown in the following corollary.

Corollary 4

Let C_m be an edge-weighted cycle on m edges with the edge weights $a_1, a_2, \ldots, a_m$ and $\sum_{i=1}^{m} a_i = mn + 1$. If $n + 1 \le a_1 \le 2n - 1$ and $2 \le a_i \le n$ for $i \in [2, m]$, then $\theta[C_m]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph.

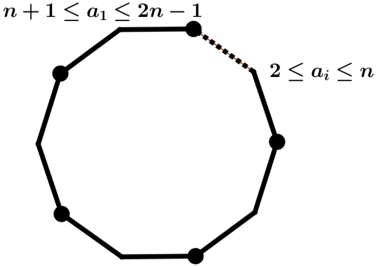

Figure 8: Illustration for Corollary 4.

Further results

For further results, we already construct a general theta-graph $\theta[F]$, based on any connected graph F.

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Theorem 3

Let p, q, n be natural numbers with $p, n \geq 2$ and $q \geq 1$. Let F be any edge-weighted connected graph on p vertices and q edges $e_1, e_2, \cdots, e_q$ with the edge weights $a_1, a_2, \cdots, a_q$, respectively where $a_i \in \mathbb{Z}^+$ and $a_i \geq 2$. Then, $\theta[F]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph if and only if

(a) $\mathrm{sum}(F) = (n-1)p + q + 1$,

(b) $\mathrm{sum}(F') < (n-1)p_1 + q_1 + 1$ for each proper connected subgraph F' of F on p_1 vertices and q_1 edges.

Open Problem

The construction of other classes of Ramsey $(C_4, K_{1,n})$-minimal graphs besides the class of theta-graphs.

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Thank you for your attention