Chiral maps of arbitrary hyperbolic type with simple underlying group

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Orientably-regular maps

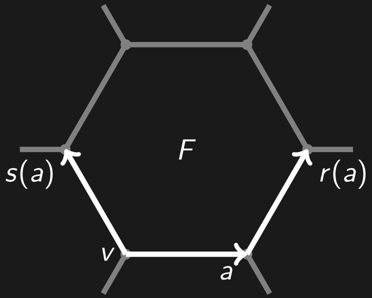

Figure: The action of r and s on the arc labelled a in an orientably-regular map $\mathcal{M}(G; r, s)$ of type $\{6, 3\}$

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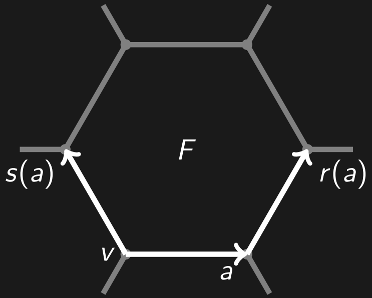
Orientably-regular maps



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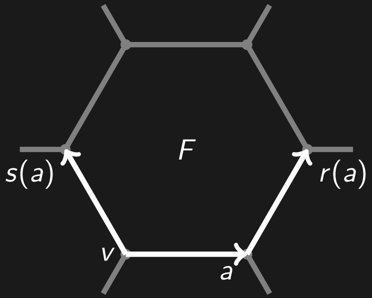

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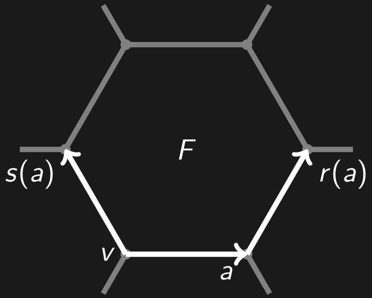

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An orientably-regular map is *reflexible* if and only if the map demonstrates reflective symmetry, that is if and only if there is an automorphism of the group G which inverts both generators, otherwise it is *chiral*.

Each chiral map has a chiral mate

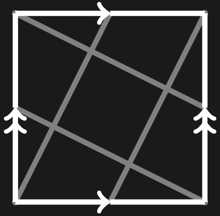

Figure: A chiral map of type $\{4, 4\}$ and its mate on the same torus with given orientation

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Also already known for when $n = 3$.

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... but sometimes we rely on:

Theorem (Jones)

Let G be a primitive permutation group of finite degree k, containing a cycle with f fixed points. Then $G \geq A_k$ if $f \geq 3$.

Lack of reflection in permutation diagram $\Rightarrow$ *chiral*?!

Lemma

Let G be such that $\mathcal{D}$ is a faithful permutation representation of $G = \langle s, t \mid s^n, t^2, (st)^m, \ldots \rangle$ which is defined on k points. Further suppose that $Aut(G) \leq S_k$. Then if the following circumstances are satisfied, this implies that the map $\mathcal{M}(G; r, s)$ is chiral.

- *There is a unique point ζ in $\mathcal{D}$ which is fixed by $s^b t$ for some non-zero b, such that ζ is not fixed by t.*
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In practice: First check that the group is alternating, and then look to see if there is any reflective symmetry in the permutation diagram...

$m = 4$ and n is odd

Let $n = 4a + 4 + i \geq 9$ where $i \in \{-1, 1\}$ and $a \geq 1$.
Define $t := (\alpha, \alpha')(\beta, \beta') \prod_{j=1}^{2a} (j, j')$ and $s := s_i$ where
$s_{-1} := (1, 2, \ldots, 2a - 1, 2a, \alpha, \alpha', \beta, 2a', 2a - 1', \ldots, 2', 1')$ and
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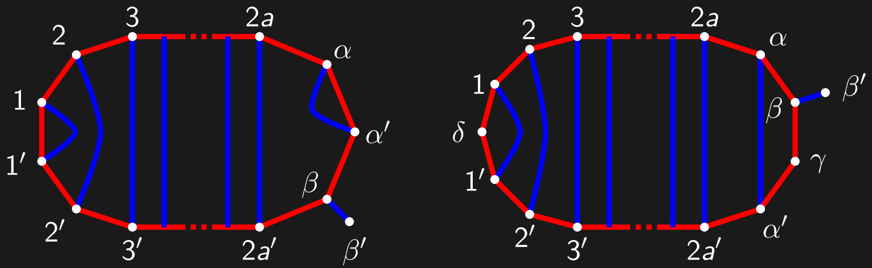
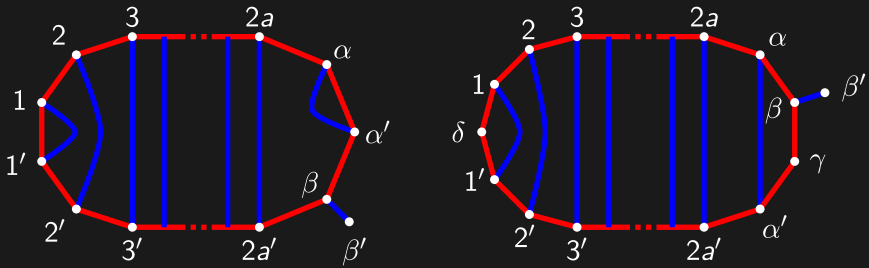


Figure: When $i = -1$ and $i = 1$ respectively

This yields chiral maps



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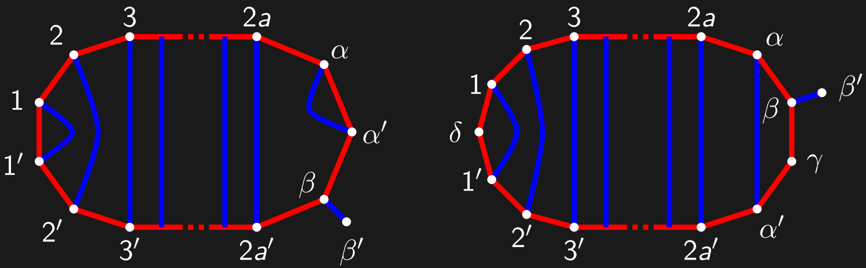

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s and t are even permutations and have the expected orders n and 2 respectively.

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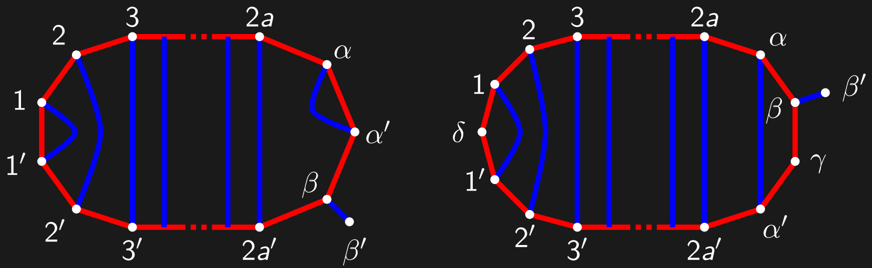

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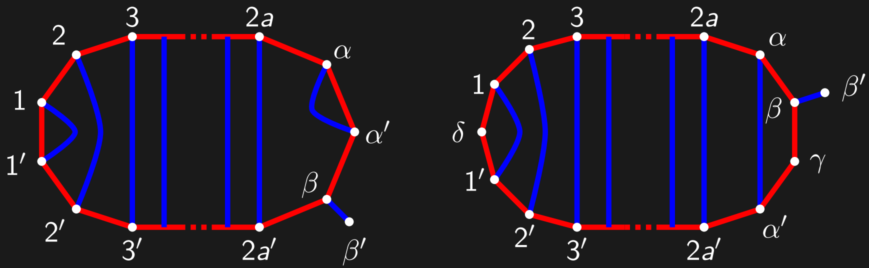

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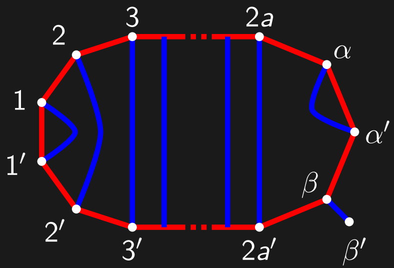

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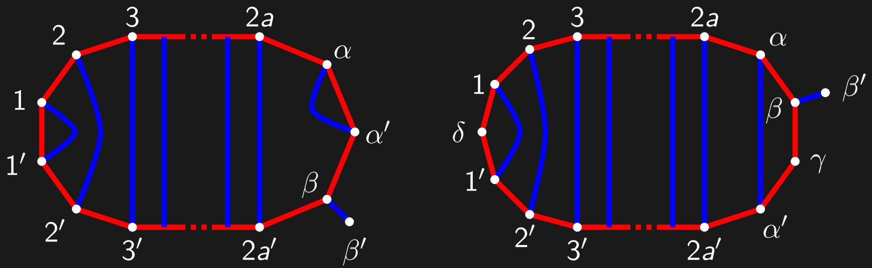

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both have order 4.

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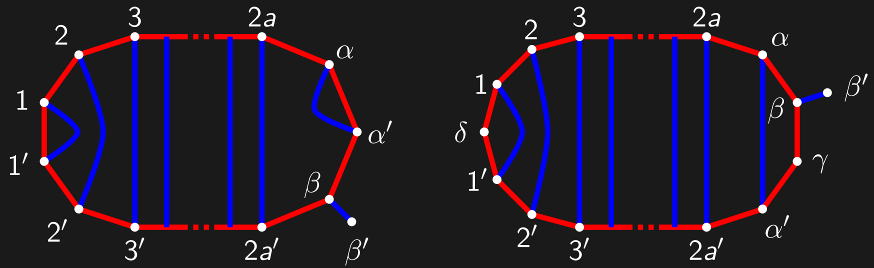

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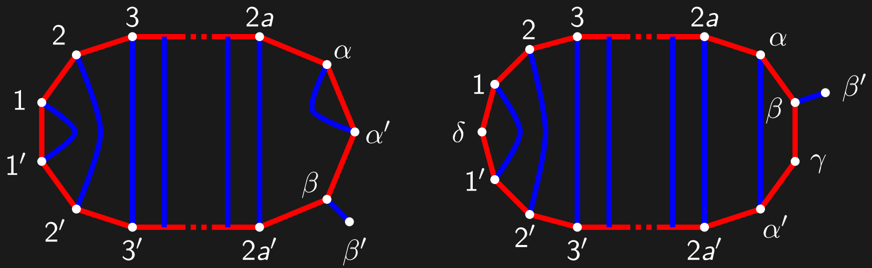

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Now consider the permutation $(s^2t)^2$ when $i = 1$ which in every case is $(\alpha, 2a, \beta, \gamma, \beta')$, a 5-cycle fixing $4a+1$ points.

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Then inspection of the corresponding diagrams in Figure 3 will yield that the map is chiral by, respectively, a-very-similar-Lemma-to-the-other, and the Lemma with $b = 3$ and $c = 2$.

□

Collect families of woodlice...

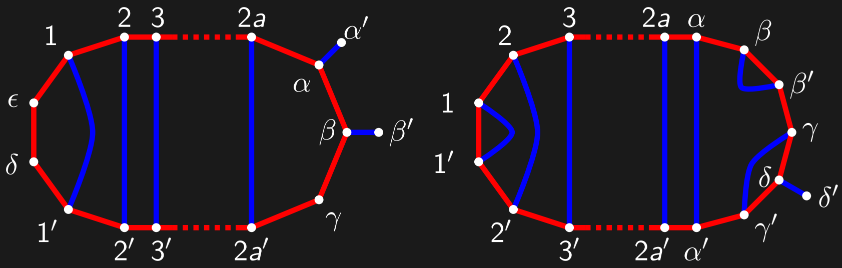

Figure: When $m = 6$

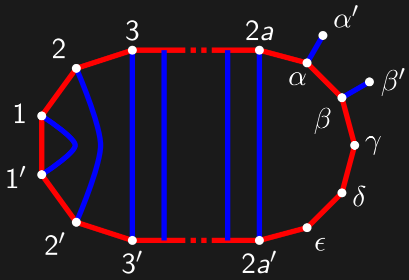

Figure: When $m = 8$

For $m \geq 10$.
Let $n = m + 4a + i$ where $i \in \{1, -1\}$ and $a \geq 0$.
For $a \geq 1$ let $t_a := \Pi_{j=1}^{a}(\alpha_j, \alpha'_j)(\beta_j, \beta'_j)$ and $r_a := \Pi_{j=1}^{a}(\alpha'_j, \beta_j)(\beta'_j, \alpha_{j+1})$, and when $a = 0$ we define t_a and r_a to be the identity. Let $t_i = (8, 9)$ when $i = -1$, and when $i = 1$ let $t_i = (\alpha_{a+1}, \alpha'_{a+1})$. Define
$r := (1, 2, \ldots, m-1, m).(1', \alpha_1).r_a$ and
$t := (1, 1')(2, 3)(4, 5)(6, 7).(m, m').t_a.t_i$.

... and other creatures ...

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$t := (1, 1')(2, 3)(4, 5)(6, 7).(m, m').t_a.t_i$.
But what do they look like...?!

... this!

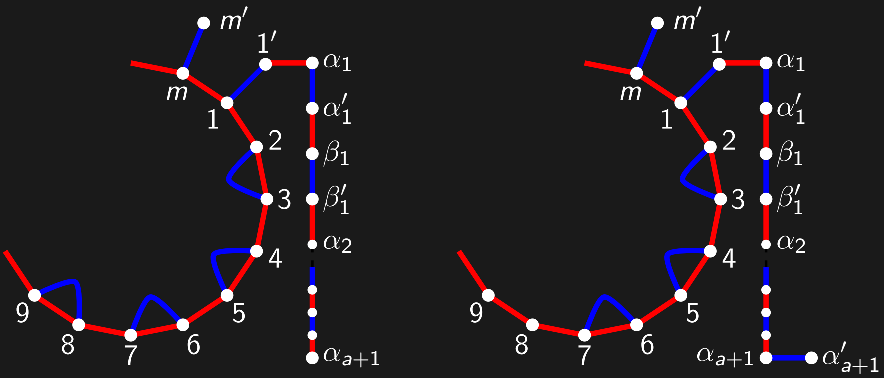

Figure: Construction when $m \geq 10$

What if even $m > n$ odd?

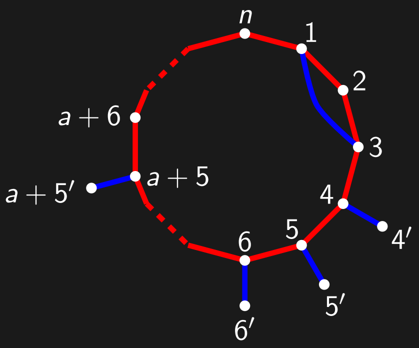

Figure: When odd $n \geq 7$

But m might be quite a lot bigger than n...

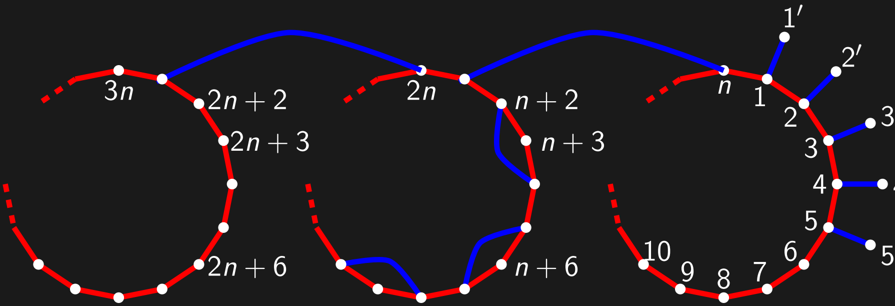

Figure: Example when $n \geq 11$

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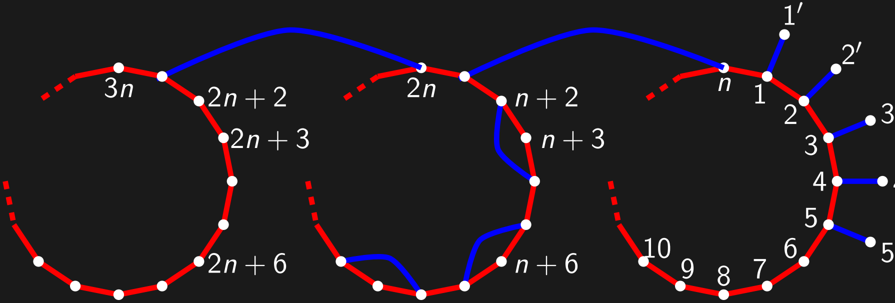

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Note: $s^2 t$ has three fixed points and a long cycle, as does its conjugate $(s^2 t)^{s^6}$. With only one fixed point in common, and since $\langle s^2 t, (s^2 t)^{s^6} \rangle$ acts transitively on all other points, we have that the stabiliser of the point labelled $n + 2$ is transitive and so the group is primitive.

What if both m and n are even?

Expand the accommodation for more woodlouse families!

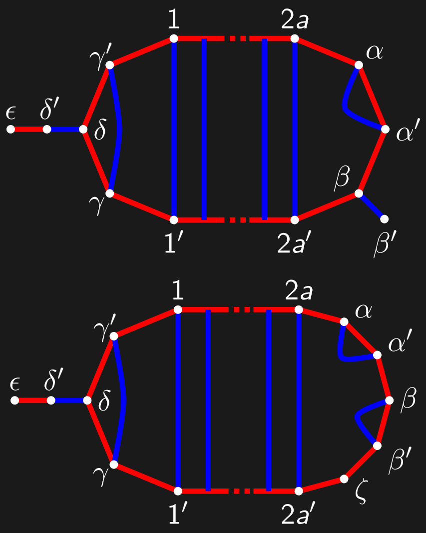


Figure: n even and $m = 4$

... and more...

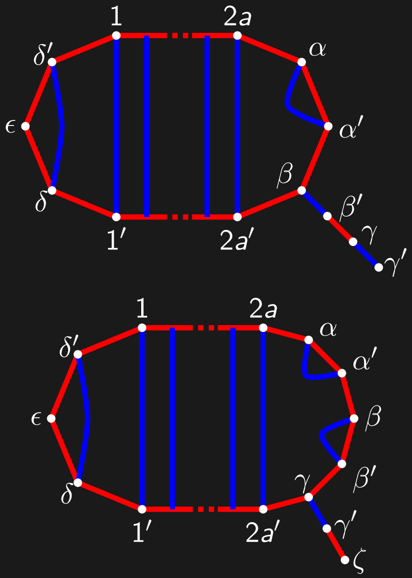
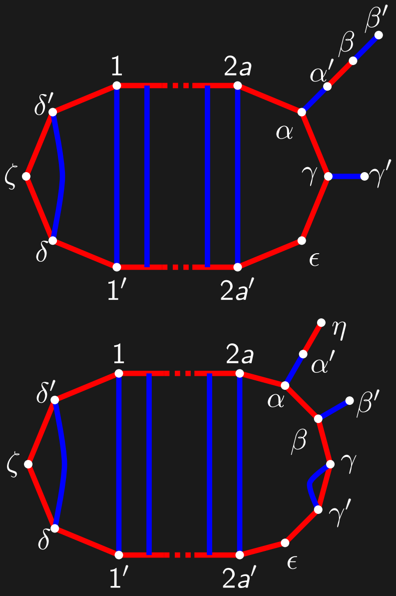


Figure: Both even and $m = 6$

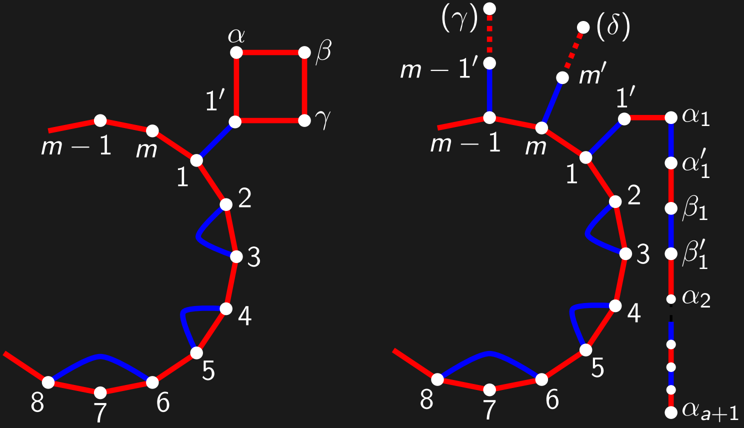
… and this completes the set of woodlice.



For $m \geq 10$.

Let $n = m + 4a + i$ where $i \in \{0, 2\}$ and $a \geq 0$.
When $m = n \equiv 0$ modulo 4 let $r = (1, 2, \ldots, m-1, m)(1', \alpha, \beta, \gamma)$ and $t = (1, 1')(2, 3)(4, 5)(6, 8)$.
Otherwise let $r = (1, 2, \ldots, m-1, m)(1', \alpha_1)r_a r_i$ and $t = (1, 1')(2, 3)(4, 5)(6, 8)(m-1, m-1')(m, m')t_a$ where, for $a \geq 1$ define $t_a := \Pi_{j=1}^{a}(\alpha_j, \alpha_j')(\beta_j, \beta_j')$ and $r_a := \Pi_{j=1}^{a}(\alpha_j', \beta_j)(\beta_j', \alpha_{j+1})$, and when $a = 0$ we define t_a and r_a to be the identity. When $i = 0$ let r_i be the identity, otherwise let $r_i = (m-1', \gamma)(m', \delta)$.

And this last pair of creatures will complete the proof*



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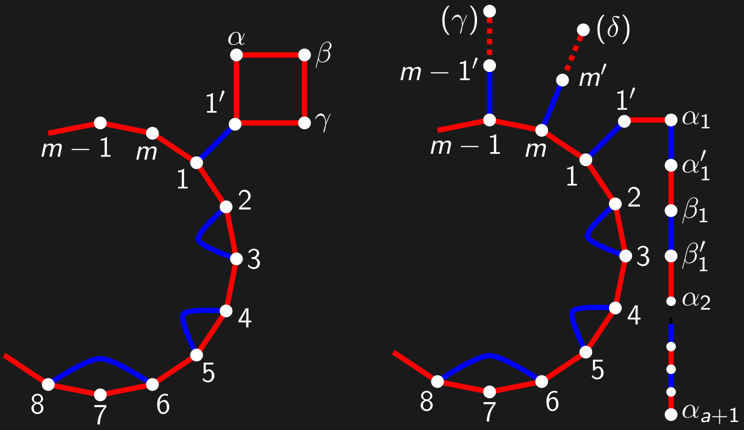

The permutation $(r^4t)^2$ fixes every point labelled with a Greek letter.

* if you include the table of small cases

Type	s	t	G
$\{4,5\}$	$(1,2,3,4,5)$	$(1,3)(2,6)$	A_6
$\{6,5\}$	$(1,2,3,4,5)(6,7,8,9,10)$	$(1,6)(2,4)(7,8)(9,10)$	A_{10}
$\{8,5\}$	$(1,2,3,4,5)(6,7,8,9,10)$	$(1,6)(2,4)$	A_{10}
$\{10,5\}$	$(1,2,3,4,5)(6,7,8,9,10)$	$(1,6)(2,4)(7,11)(8,12)$	A_{12}
$\{12,5\}$	$(1,2,3,4,5)(6,7,8,9,10)$	$(1,6)(2,4)(7,11)(8,12)(9,13)(10,14)$	A_{14}
$\{6,6\}$	$(1,2,3,4,5,6)(7,8,9,10,11,12)$	$(1,7)(2,5)(6,13)(8,12)(9,14)(10,15)$	A_{15}
$\{4,7\}$	$(1,2,3,4,5,6,7)$	$(1,5)(2,3)(4,9)(7,8)$	A_9
$\{6,7\}$	$(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$	$(1,3)(4,6)(7,8)(9,13)(10,15)(11,16)$	A_{16}
$\{10,7\}$	$(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$	$(1,8)(2,4)(5,6)(9,10)$	A_{14}
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$\{14,7\}$	$(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$	$(1,8)(7,9)(10,15)(11,16)$	A_{16}
$\{14,9\}$	$(1,2,3,4,5,6,7,8,9)(10,11,12) := S_9$	$(1,10)(2,12)(3,13)(4,14)(5,15)(6,16)$	A_{16}
$\{16,9\}$	$S_9(13,17,18)$	$(1,10)(2,12)(3,13)(4,14)(5,15)(6,16)$	A_{18}
$\{18,9\}$	$S_9(13,17,18)(14,19,20)$	$(1,10)(2,12)(3,13)(4,14)(5,15)(6,16)$	A_{20}
$\{20,9\}$	$S_9(13,17,18)(14,19,20)(15,21,22)$	$(1,10)(2,12)(3,13)(4,14)(5,15)(6,16)$	A_{22}

Table: Filling gaps - the missing types

* and trust me that the ‘small’ infinite cases work...

...which they do.

What just happened?

Theorem

Given a hyperbolic type $\{m, n\}$, there exists a chiral map of that type with alternating automorphism group A_k, for some degree k.