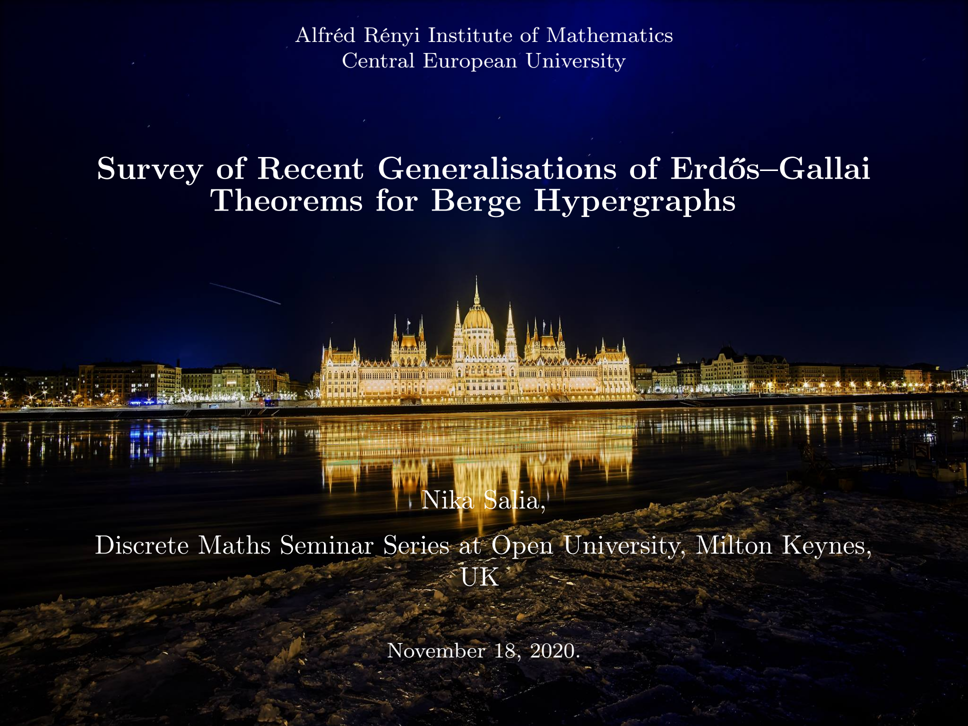


Alfréd Rényi Institute of Mathematics
Central European University

Survey of Recent Generalisations of Erdős–Gallai Theorems for Berge Hypergraphs

Nika Salia,

Discrete Maths Seminar Series at Open University, Milton Keynes, UK

November 18, 2020.

Extremal Graph Theory

B. Bollobás, Extremal Graph Theory

- Extremal graph theory is a branch of graph theory developed and loved by Hungarians.
- Its study as a subject was initiated by P. Turán in 1940.
- Several other extremal results including a special case of Turán's theorem had been proved many years earlier.
- The main exponent has been Paul Erdős who, through his many papers and lectures, as well as unaccountably many problems, has virtually created the subject.

Why is it interesting?

There is a pleasure sure, In being mad, which none but madmen know!
John Dryden, 'The Spanish Friar'.

Theorem (Turán, 1940)

If G is a graph of order n, without a clique of size $r+1$, then

$$e(G) \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}$$

Theorem (Mantel, 1907)

If G is a triangle free graph of order n, then

$$e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

Remark

The equality holds iff $r|n$ and G is the complete r-partite graph with equal classes.

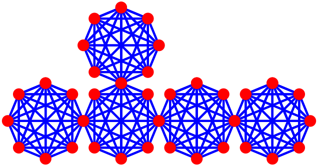
Theorem (Erdős-Gallai, 1959)

If G is a graph of order n, without a cycle of length at least ℓ, then

$$e(G) \leq \frac{(\ell-1)(n-1)}{2}$$

Remark

The equality holds iff $\ell - 2 | n - 1$ and G is the union of $\frac{n-1}{\ell-2}$ disjoint cliques of size $\ell - 1$ sharing a vertex.



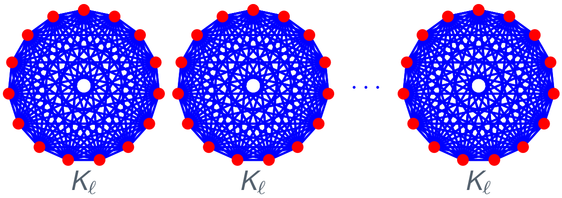
Theorem (Erdős-Gallai, 1959)

If G is a graph of order n, without a path of length ℓ, then

$$e(G) \leq \frac{\ell - 1}{2} n$$

Remark

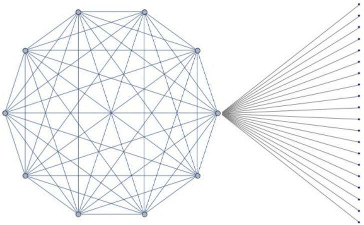
The equality holds iff $\ell | n$ and G is the disjoint union of $\frac{n}{\ell}$ cliques of size ℓ.

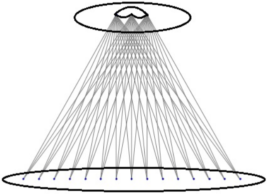


Theorem (Kopylov 1977, Ballister, Győri, Lehel, Schelp 2008)

*If G is a **connected** graph of order n, without a path of length k, $n > k \geq 3$, then number of edges $e(G)$ is at most*

$$\max\left\{\binom{k-1}{2} + n - k + 1, \binom{\lceil\frac{k+1}{2}\rceil}{2} + \left\lfloor\frac{k-1}{2}\right\rfloor\left(n - \left\lceil\frac{k+1}{2}\right\rceil\right)\right\}$$

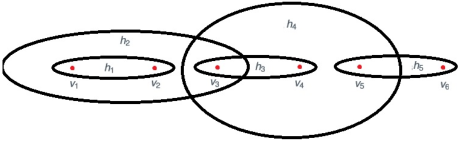




There are many different ways to define paths in hypergraphs. One of the most basic/general definition is due to **Berge.**

Definition (Berge Path of Length k)

A Berge path of length k in a Hypergraph $\mathcal{H}$, is an ordered $k+1$ distinct vertices $v_1, v_2, \ldots, v_{k+1}$ and ordered k distinct hyperedges $h_1, h_2, \ldots, h_k$ such that for $1 \leq i \leq k$, v_i and $v_{i+1} \in h_i$.



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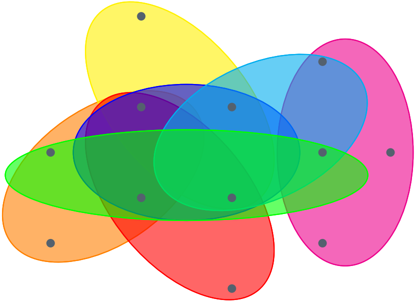

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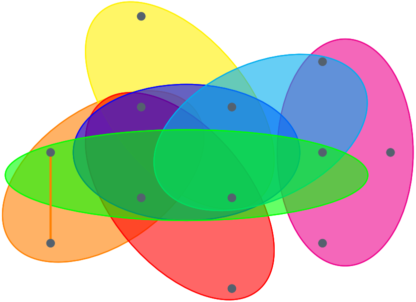

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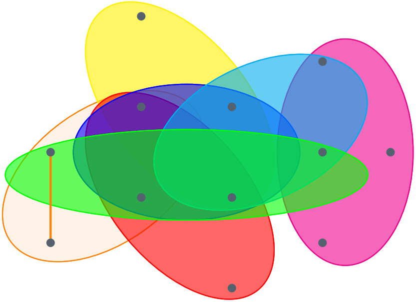

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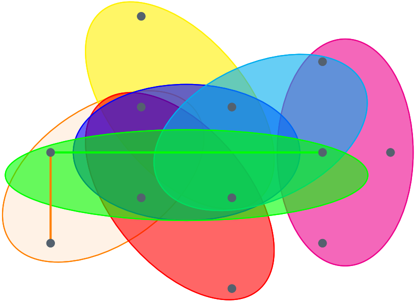

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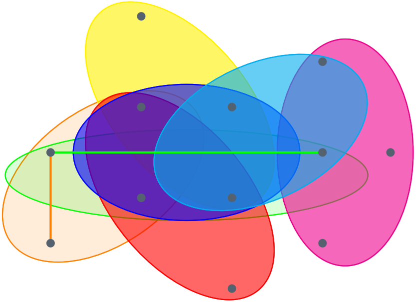

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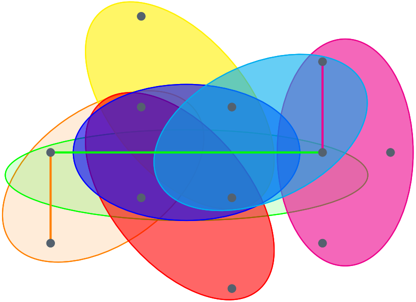

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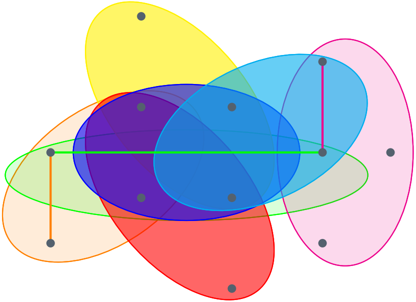

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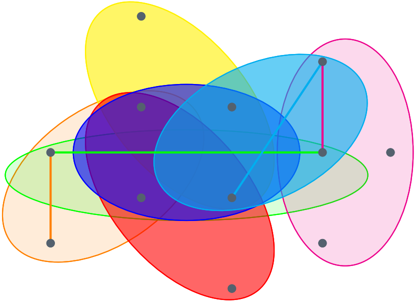

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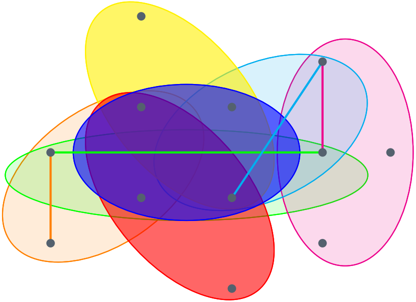

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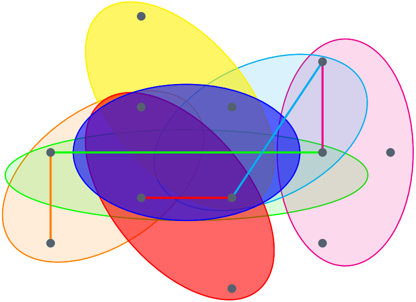

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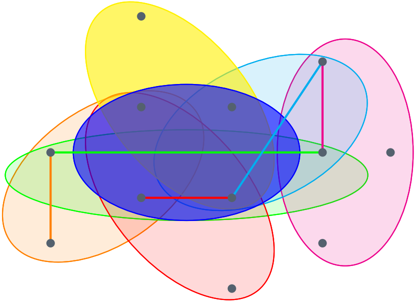

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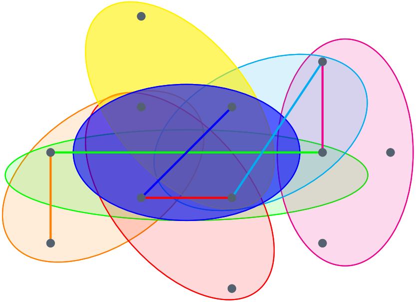

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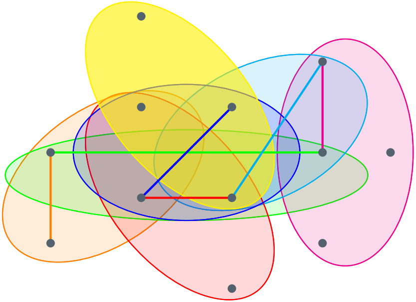

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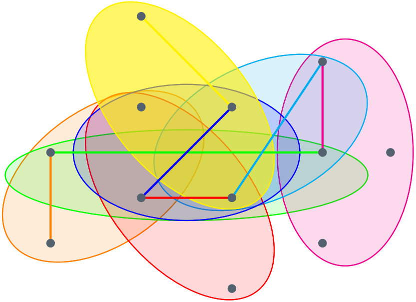

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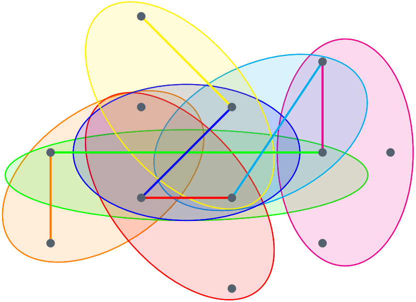

Cycles in the Hypergraph?

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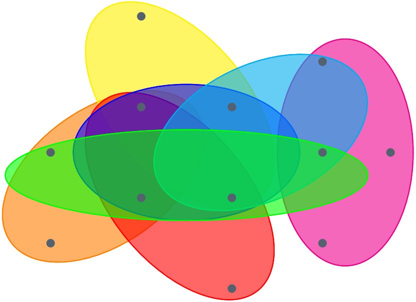

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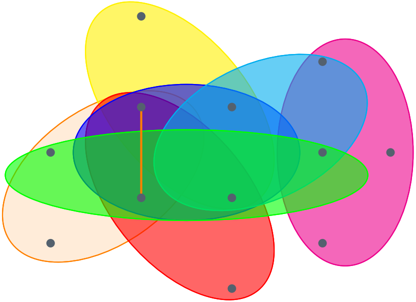

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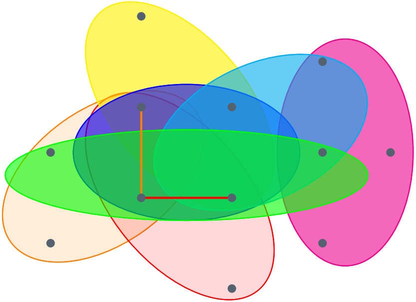

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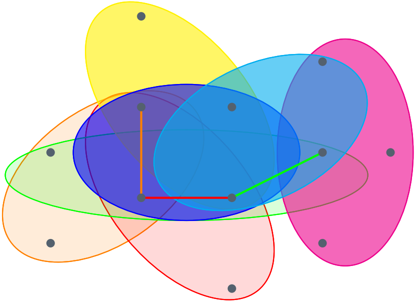

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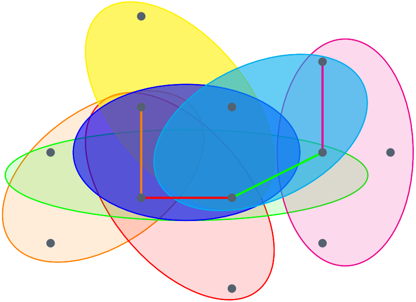

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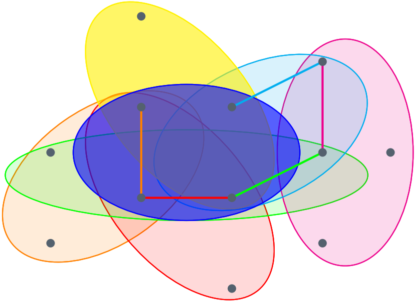

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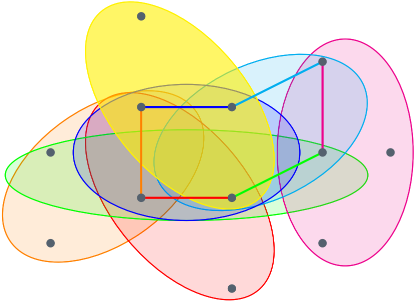
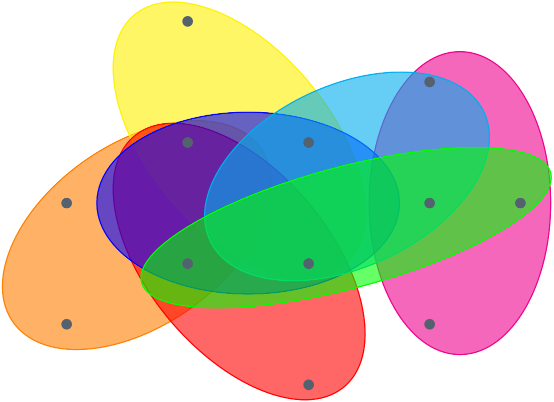


Does this hypergraph contain a Berge cycle of length at least 7?



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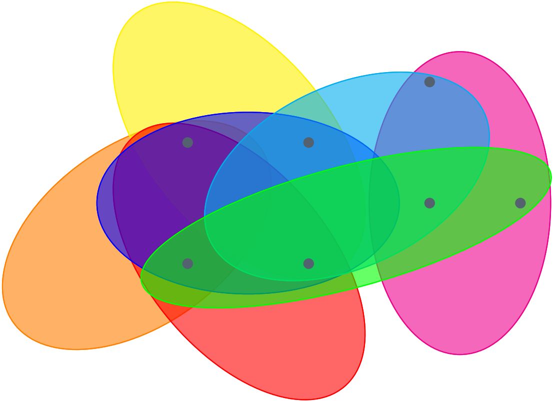

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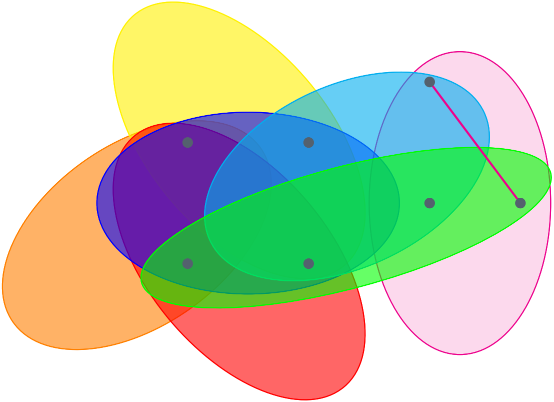

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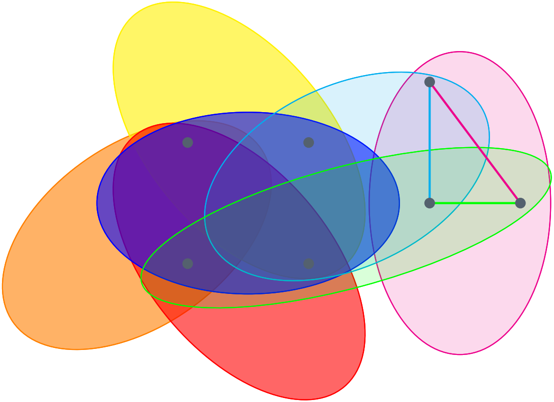
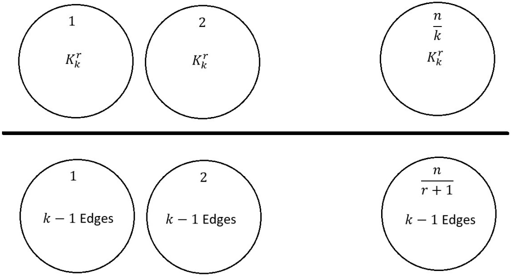


Theorem (Győri, Katona, Lemons, 2016 Davoodi, Győri, Methuku, Tompkins, 2018)

If $\mathcal{H}$ is an r-uniform hypergraph of order n without a Berge path of length k, then

- *If $k \geq r+1 > 2$ we have $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$*
- *If $r \geq k > 2$ we have $e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}$*



Hypergraph Version, Berge Cycles, $k \geq r + 3$

Erdős-Gallai

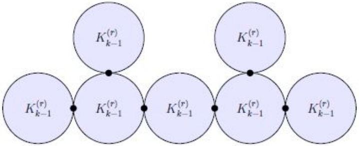
Theorem (Füredi, Kostochka, Luo, 2018)

If $\mathcal{H}$ is an r-uniform hypergraph of order n, without a Berge cycle of length at least k, $k \geq r + 3$, then

$$e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$$

Remark

The equality holds iff $k-2|n-1$ and $\mathcal{H}$ is the union of $\frac{n-1}{k-2}$ disjoint r-uniform cliques of size $k-1$ sharing a vertex



Hypergraph Version, Berge Cycles, k=r+2

Erdős-Gallai

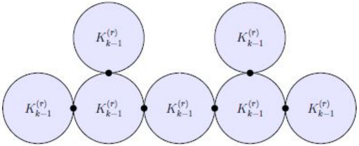
Theorem (E., G., M., S., T., Z., 2018)

If $\mathcal{H}$ is an r-uniform hypergraph of order n, without a Berge cycle of length at least $r+2$, then

$$e(\mathcal{H}) \leq \frac{(n-1)(r+1)}{r}$$

Remark

The equality holds iff $r|n-1$ and $\mathcal{H}$ is the union of $\frac{n-1}{r}$ disjoint r-uniform cliques of size $k-1$ sharing a vertex



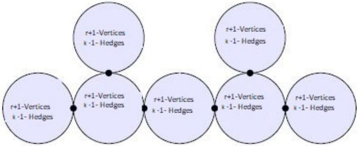
Theorem (E., G., M., S., T., Z., 2020)

If $\mathcal{H}$ is an r-uniform hypergraph of order n, without a Berge cycle of length at least $r+1$, then

$$e(\mathcal{H}) \leq n-1$$

Remark

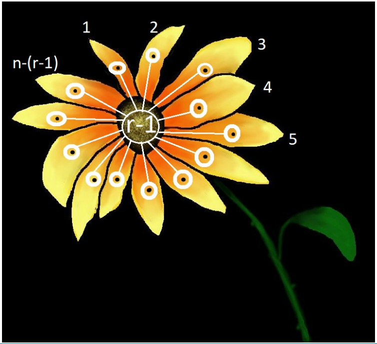
The equality holds iff $r|n-1$...



Theorem (G., L., S., Z., 2018)

If $\mathcal{H}$ is an r-uniform hypergraph of order n, without a Berge cycle of length r, then

$$e(\mathcal{H}) \leq \max\left(n-(r-1), \frac{(n-1)(r-1)}{r}\right)$$



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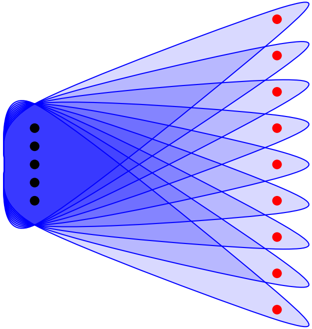


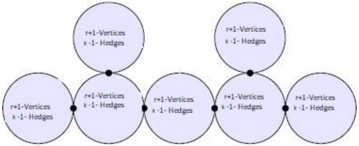
Theorem (Kostochka, Luo, G., L., S., Z. 2020)

If $\mathcal{H}$ is an r-uniform hypergraph of order n, without a Berge cycle of length at least k, $k \leq r-1$, then

$$e(\mathcal{H}) \leq \frac{n-1}{r}(k-1)$$

Remark

The equality holds iff $r|n-1$ and $\mathcal{H}$ is the union of size $r+1$ vertex sets, containing $k-1$ hyperedges, sharing a vertex.



Connected Hypergraph Version

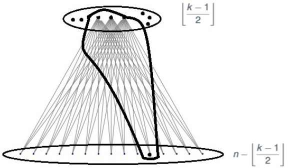
Erdős-Gallai

Theorem (Győri, Salia, Zamora 2020+ and Füredi, Kostochka, Luo 2020+)

Let $\mathcal{H}_{n,k}$ be a largest r-uniform, $k \geq 2r + 13 \geq 18$, connected hypergraph of order $n > n_{k,r}$, without a Berge path of length k, then

- $e(\mathcal{H}_{n,k}) = |\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor}|$, *if k is odd, and*
- $e(\mathcal{H}_{n,k}) = |\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}|$, *if k is even.*

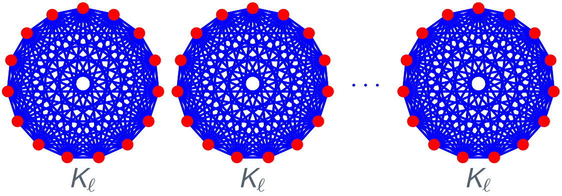
Depending on the parity of k, the unique extremal hypergraph is $\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor}$ or $\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}$.



Conjecture (Erdős and Sós, 1984)

If G is a graph of order n, without a fixed Tree T with ℓ edges, then

$$e(G) \leq \frac{\ell - 1}{2} n$$



Theorem (Gerbner, Methuku, Palmer, 2020)

If the Erdős-Sós conjecture holds for a tree T with ℓ edges and all of its sub-trees and $\ell > r+1$, then for every r-uniform hypergraph with $n-$vertices and no Berge tree T we have

$$e(\mathcal{H}) \leq \frac{n}{\ell}\binom{\ell}{r}$$

Remark

The equality holds iff $\ell | n$ and $\mathcal{H}$ is the disjoint union of $\frac{n}{\ell}$ r-cliques of size ℓ.

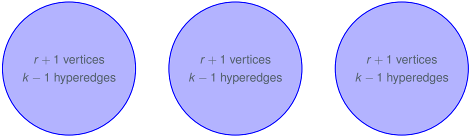
Berge Trees $r \geq \ell(\ell-2)$

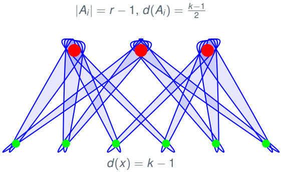
Erdős and Sós

Theorem (G,S,T,Z,2020+)

Let n, ℓ, r be positive integers and let T be a ℓ-edge tree which is not a star, then we have for all $r \geq \ell(\ell-2)$, every n vertex $r-$uniform hypergraph $\mathcal{H}$, without a copy of Berge-T

$$e(\mathcal{H}) \leq \frac{n(\ell-1)}{r+1}.$$





An open problem, I am particularly interested in

Question (Kostochka, Lavrov, Luo, Zirlin)

Let $\mathcal{H}$ be an $n-$vertex, NON-uniform, hypergraph. We have

- *For every $A \subseteq V(\mathcal{H})$, $(|A| \geq 2)$, the number of hyperedges incident with at least two vertices from A is at least $|A|$;*

Then for every $A \subseteq V(\mathcal{H})$, $(|A| \geq 2)$, there is a Berge cycle with the base vertices on A.


Thank You!
Please Ask Me Questions?
Nika Salia