THE METRIC DIMENSION OF (EDGE) COMB PRODUCT GRAPHS

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TRANSITION
SLIDE
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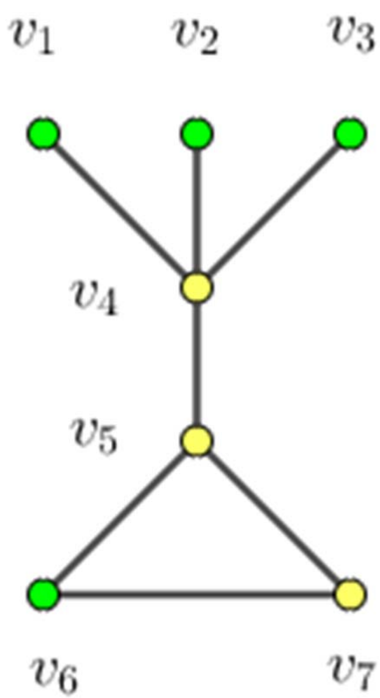
01

DEFINITION

- Let G be a simple, connected, and finite graph.
- The distance between two distinct vertices $u, v \in V(G)$, denoted by $d_G(u, v)$, is the length of a shortest path from u to v in G.
- Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$.
- For $v \in V(G)$, a representation of v with respect to W is defined as k-tuple $r(v|W) = (d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_k))$.
- A vertex set W resolves a graph G if every two distinct vertices $x, y \in V(G)$ satisfies $r(x|W) \neq r(y|W)$.
- The resolving set of G with minimum cardinality is called a basis of G, and we called its cardinality as the metric dimension of G, denoted by $\beta(G)$.

EXAMPLE

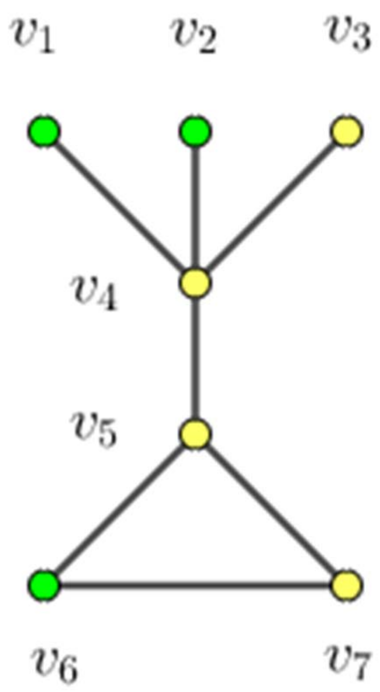


$W_1 = \{v_1, v_2, v_3, v_6\}$

$r(v_1|W_1) = \{0, 2, 2, 3\}$
$r(v_2|W_1) = \{2, 0, 2, 3\}$
$r(v_3|W_1) = \{2, 2, 0, 3\}$
$r(v_4|W_1) = \{1, 1, 1, 2\}$
$r(v_5|W_1) = \{2, 2, 2, 1\}$
$r(v_6|W_1) = \{3, 3, 3, 0\}$
$r(v_7|W_1) = \{3, 3, 3, 1\}$

W_1 is a resolving set of G.

EXAMPLE



$W_2 = \{v_1, v_2, v_6\}$

$$
\begin{aligned}
r(v_1|W_2) &= \{0, 2, 3\} \\
r(v_2|W_2) &= \{2, 0, 3\} \\
r(v_3|W_2) &= \{2, 2, 3\} \\
r(v_4|W_2) &= \{1, 1, 2\} \\
r(v_5|W_2) &= \{2, 2, 1\} \\
r(v_6|W_2) &= \{3, 3, 0\} \\
r(v_7|W_2) &= \{3, 3, 1\}
\end{aligned}
$$

W_2 is a resolving set of G.

Since there is no resolving set with 2 vertices, then $\beta(G) = 3$

KNOWN RESULTS

G. Chartrand, L. Eroh, M.A. Johnson, O.R. Oellermann (2000)

Theorem

Let G be a connected graph of order $n \geq 2$ and diameter d.

$$f(n,d) \leq \beta(G) \leq n-d$$

where $f(n,d)$ is the least positive integer k for which $k + d^k \geq n$.

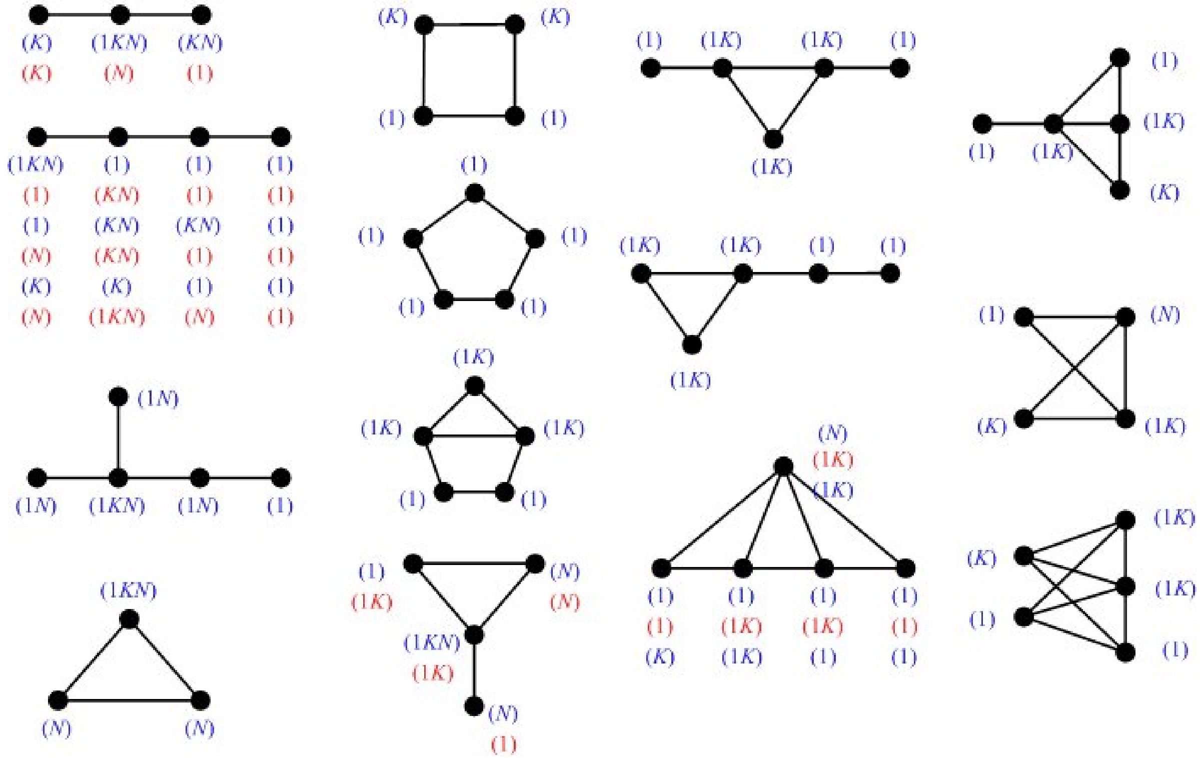
Moreover, they also characterized all graphs G of order n with metric dimension $1, n-1, n-2$.

KNOWN RESULTS

M. Jannesari, B. Omoomi (2014)

Theorem

Let G^* be a twin graph of a connected graph G of order $n \geq 4$. Then $\beta(G) = n - 3$ if and only if a twin graph of G whose type is stated in the follwing figure.



KNOWN RESULTS

S. Khuller, B. Raghavachari, A. Rosenfeld (1996)

Theorem

Let T be a tree which is not a path. Then

$$\beta(T) = \sum_{v \in V(T);\, l_v > 1} (l_v - 1)$$

where l_v is the number of components of $T \setminus v$ which is not a path with length at least 0.

KNOWN RESULTS

I. Javaid, M.T. Rahman, K. Ali (2008)

Theorem

For $n \geq 5$, let $H_{4,n}$ be a Harary regular-4 graph. Then

$$\beta(H_{4,n}) = \begin{cases} 3, & \text{for } n \equiv 0,2,3 \pmod 4, \\ 4, & \text{otherwise.} \end{cases} .$$

C. Grigorious, P. Manuel, M. Miller, B. Rajan, S. Stephen (2014)

Theorem

Let G be a Harary graph $H(r,n)$ such that r is odd, n is even and $n \equiv 2j + 2i \pmod 4$, $2 \leq i \leq j+1$. Then $\beta(G) = j+2$ where $j = \dfrac{r-1}{2}$.

KNOWN RESULTS

P.S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang (2003)

Theorem

If G' is a graph obtained by adding a pendant edge to a nontrivial connected graph G, then

$$\beta(G) \leq \beta(G') \leq \beta(G) + 1.$$

C. Poisson, P. Zhang (2002)

Theorem

Let T be a tree graph of order at least 3 and e be an edge in $\overline{T}$. Then

$$\beta(T) - 2 \leq \beta(T + e) \leq \beta(T) + 1.$$

KNOWN RESULTS

J. Caceres, C. Hernando, M. Mora, M.L. Puertas, I.M. Pelayo, C. Seara (2005)

Theorem

Let G and H be connected graphs with order at least 2. Then

$$\beta(G) + \beta(H) \leq \beta(G + H).$$

Theorem

For $n \notin \{1, 2, 3, 6\}$, $\beta(P_n + K_1) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

KNOWN RESULTS

J. Caceres, C. Hernando, M. Mora, M.L. Puertas, I.M. Pelayo, C. Seara, D.R. Wood (2007)

Theorem

For every graph G and $n \geq 1$,

$$\beta\left(G \Box K_n\right) \leq \max\left\{n-1, 2 \cdot \beta\left(G\right)\right\}.$$

Theorem

For every graph G,

- $\beta\left(G\right) \leq \beta\left(G \Box P_n\right) \leq \beta\left(G\right) + 1$ with $n \geq 2$
- $\beta\left(G\right) \leq \beta\left(G \Box C_n\right) \leq \begin{cases} \beta\left(G\right)+1, & \text{if } n \text{ is odd;} \\ \beta\left(G\right)+2, & \text{if } n \text{ is even.} \end{cases}$

KNOWN RESULTS

H. Iswadi, E.T. Baskoro, R. Simanjuntak (2011)

Theorem

Let G be a connected graph of order n and H be a graph with $|V(H)| \geq 2$. Then

$$\beta(G \odot H) = \begin{cases} n \cdot (\beta(H + K_1) - 1), & \text{if } H \text{ contains a dominant vertex,} \\ n \cdot \beta(H + K_1), & \text{otherwise.} \end{cases}$$

I.G. Yero, D. Kuziak, J.A. Rodríguez-Velázquez (2011)

Theorem

Let G be a connected graph of order n and H be a graph with $|V(H)| \geq 2$. Then

$$\beta(G \odot H) = \begin{cases} n \cdot \beta(H), & \text{if } diam(H) \leq 2, \\ n \cdot \beta(H + K_1), & \text{if } diam(H) \geq 6 \text{ or } H \text{ is a cycle.} \end{cases}$$

02

METRIC DIMENSION

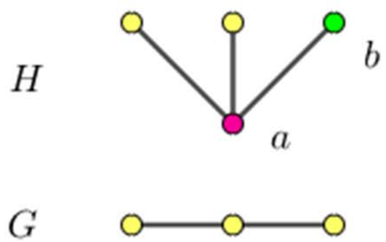
COMB PRODUCT GRAPHS

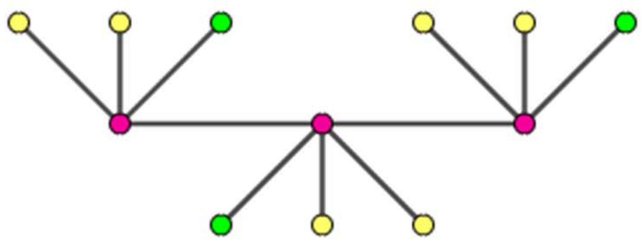
SWS, N. Mardiana, I.A. Purwasih (2017)

COMB PRODUCT

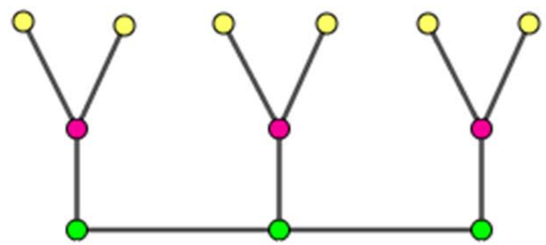
Definition

Let G and H be two connected graphs. Let o be a vertex of H. The comb product between G and H, denoted by $G \rhd_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identify the i-th copy of H at the vertex o with the i-th vertex of G.





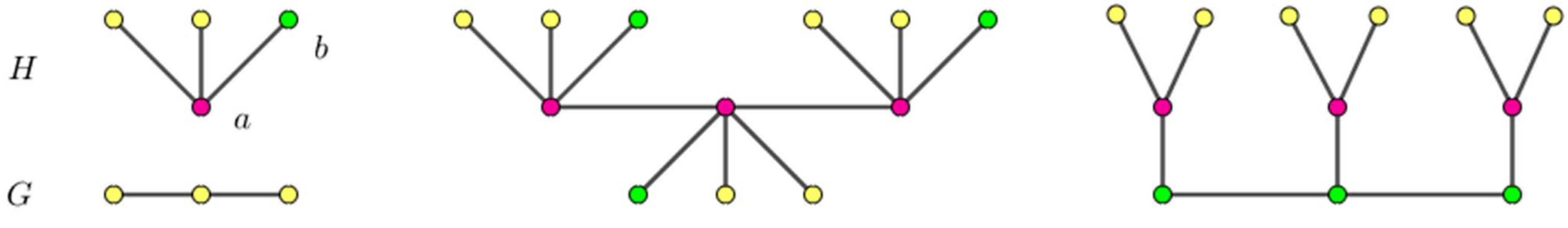
$G \rhd_a H$



$G \rhd_b H$

COMB PRODUCT

- $V(G \rhd_o H) = \{(x,u) | x \in V(G), u \in V(H)\}$
- $(x,u)(y,v) \in E(G \rhd_o H)$ if $(x = y$ and $uv \in E(H))$ or $(xy \in E(G)$ and $u = v = o)$
- For $o \in V(H)$ and $x \in V(G)$, we define $G(o) = \{(x,o) | x \in V(G)\}$ and $H(x) = \{(x,u) | u \in V(H)\}$



$G \rhd_a H$ $\qquad$ $G \rhd_b H$

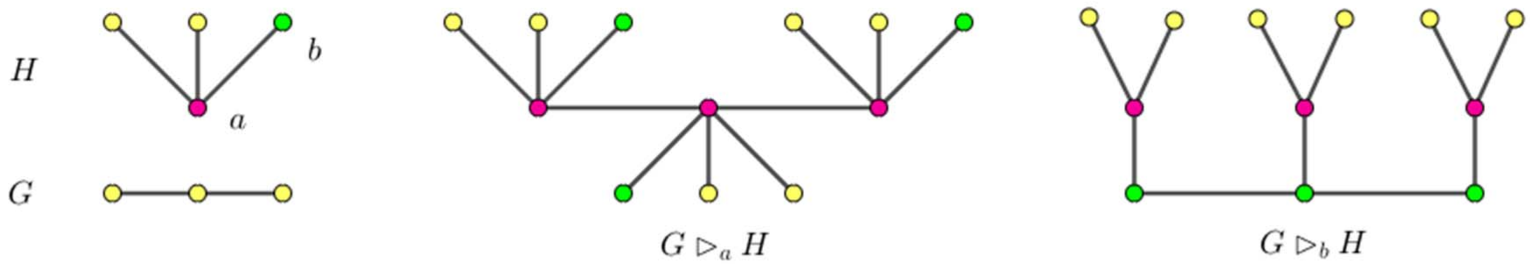
COMB PRODUCT

Lemma

Let G and H be connected graphs of order at least 2. Let H be satisfying one of two conditions below.

1. H is not a path; or
2. H is a path and the degree of the vertex o is 2.

Then there exist two distinct vertices $x, y \in V(H) \backslash \{o\}$ such that $d_{G \rhd_o H}((a,x),(a,o)) = d_{G \rhd_o H}((a,y),(a,o))$ for every vertex $a \in V(G)$.

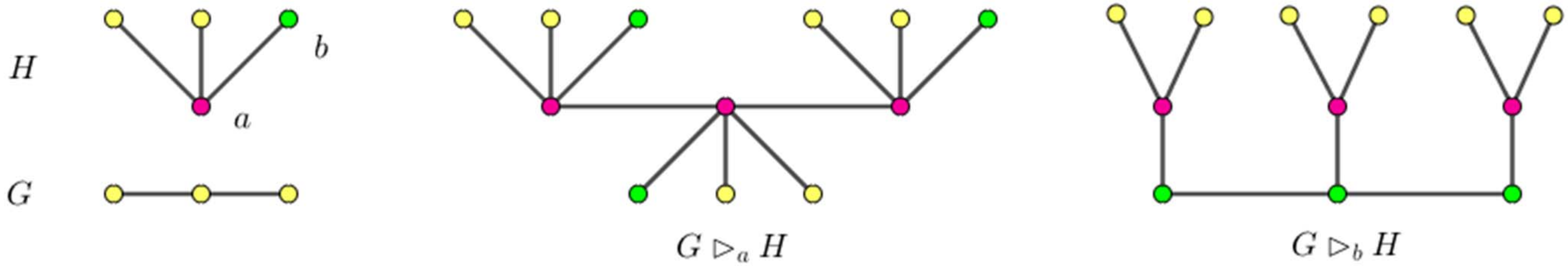


COMB PRODUCT

Lemma

Let G and H be connected graphs of order at least 2. Let H be not a path or H be a path where the vertex o is not a leaf. Let W be a basis of $G \rhd_o H$. For any vertex $a \in V(G)$, if $S(a) = W \cap H(a)$, then $S(a) \neq \emptyset$. Moreover, if B is a basis of H, then $|S(a)| \leq |B|$.

$$\beta(G \rhd_o H) \leq |V(G)| \cdot \beta(H).$$

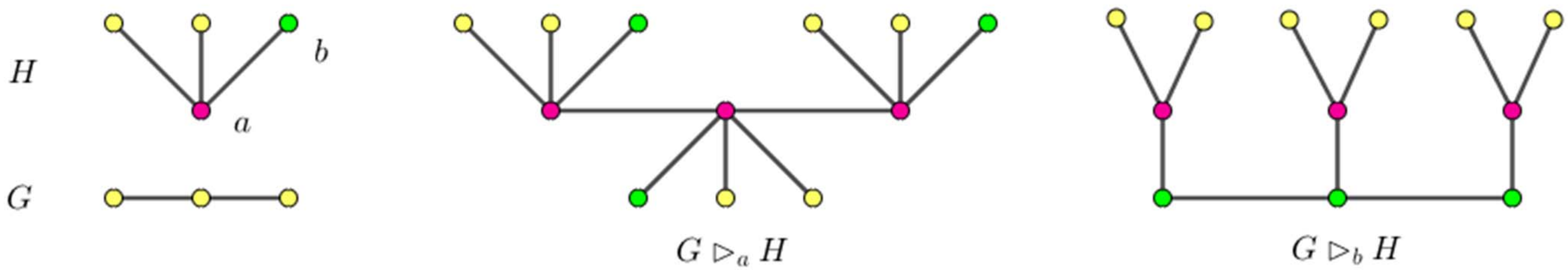


COMB PRODUCT

Lemma

Let G and H be connected graphs of order at least 2. Let H be not a path or H is a path where the vertex o is not a leaf. Let W be a basis of $G \rhd_o H$. For any vertex $a \in V(G)$, if $S(a) = W \cap H(a)$, then $|S(a)| \geq \beta(H) - 1$.

$$\beta(G \rhd_o H) \geq |V(G)| \cdot (\beta(H) - 1).$$

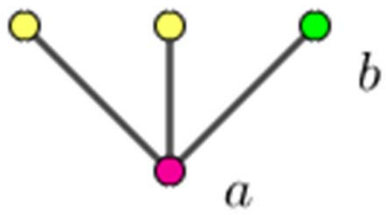


COMB PRODUCT

Lemma

Let G and H be connected graphs of order at least 2. Let H be not a path or H is a path where the vertex o is not a leaf. Then

$$|V(G)| \cdot (\beta(H) - 1) \leq \beta(G \triangleright_o H) \leq |V(G)| \cdot \beta(H)$$



Let $o \in V(H)$ be an identifying vertex. We say that a graph H is of:

- type $\mathcal{A}_o$ if there exists there exists a basis of H containing o.
- type $\mathcal{B}_o$ if H is not of type $\mathcal{A}_o$.

COMB PRODUCT

- type $\mathcal{A}_o$ if there exists there exists a basis of H containing o.
- type $\mathcal{B}_o$ if H is not of type $\mathcal{A}_o$.

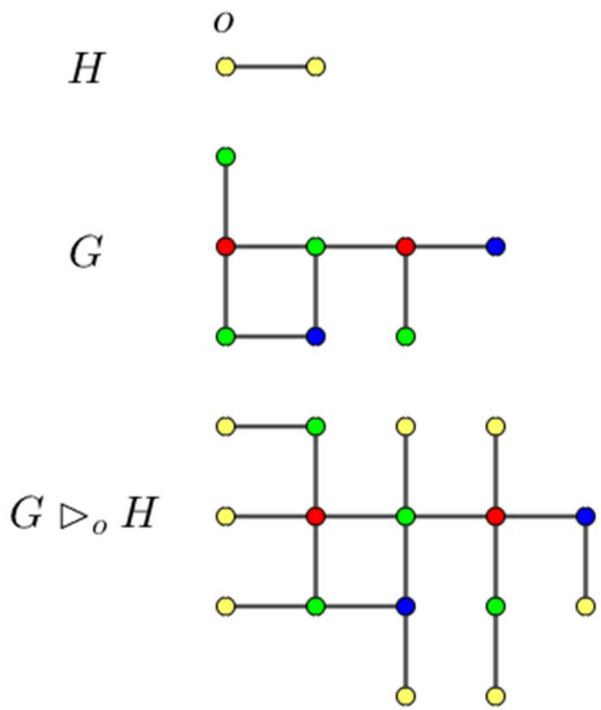
Theorem

Let G and H be connected graphs of order at least 2. Let H be not a path or H is a path where the vertex o is not a leaf. Then

$$\beta(G \rhd_o H) = \begin{cases} |V(G)| \cdot (\beta(H) - 1), & \text{if } H \text{ is of type } \mathcal{A}_o, \\ |V(G)| \cdot \beta(H), & \text{if } H \text{ is of type } \mathcal{B}_o. \end{cases}$$

COMB PRODUCT

Let G be a connected graph and H be a path where o is an end point of H.



- Let v be a vertex of G
- A branch of G at v is defined as a maximal subgraph of G which is isomorphic to a tree and containing v as an end point.
- A branch of v which is isomorphic to a path is called a path branch of v.
- If v contains at least two path branches, then v is called a stem of G.

Theorem

Let G be a connected graph containing $p \geq 1$ stems.
If the vertex o is a leaf of P_n, then $\beta(G \rhd_o P_n) \geq \beta(G) + p$.

03

METRIC DIMENSION

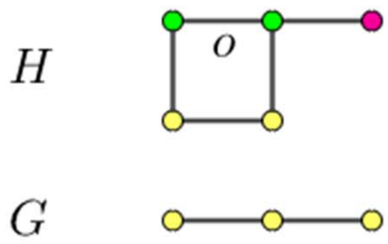
EDGE-COMB PRODUCT GRAPHS

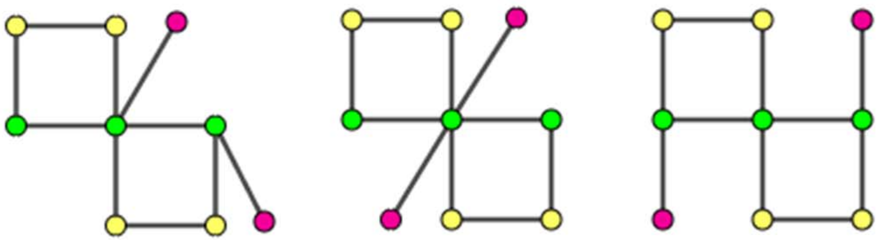
SWS, E. Suwastika, P.E. Putri (2024+)

EDGE-COMB PRODUCT

Definition

Let G and H be two connected graphs. Let o be an edge of H. The edge-comb product between G and H, denoted by $G \trianglerighteq_o H$, is a graph obtained by taking one copy of G and $|E(G)|$ copies of H and identify the i-th copy of H at the edge o with the i-th edge of G.

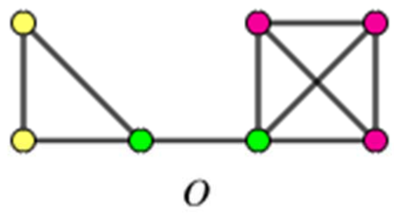




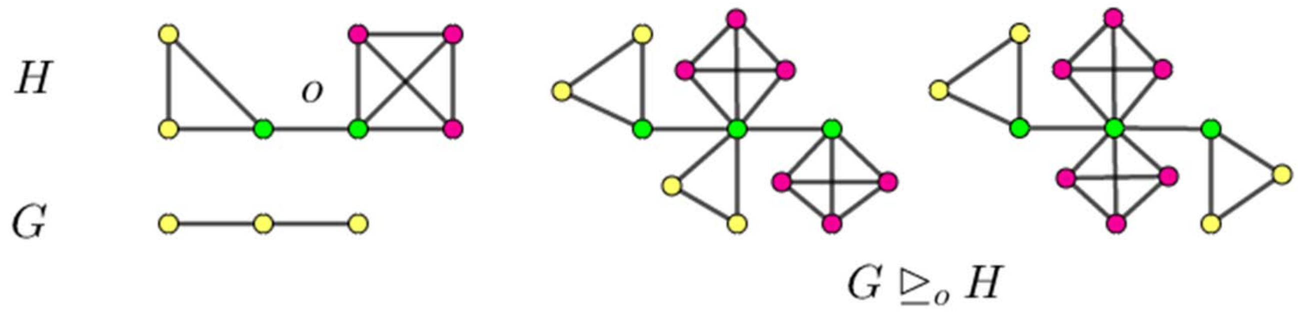
$G \trianglerighteq_o H$

EDGE-COMB PRODUCT

Let G and H be a connected graphs and o is a bridge in H where $H \setminus o$ does not contain a path as its component.



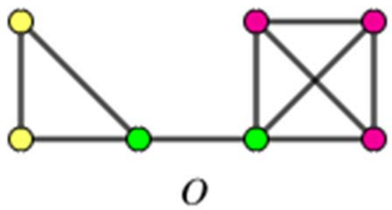
- Let $o = o_1 o_2$ where $o_1, o_2 \in V(H)$.
- Let $H \setminus \{o\} = H_1 \cup H_2$ the vertex o_i is contained in H_i.
- For $v \in V(G)$, let $l_i(v)$ as the number of vertex o_i identified to v.
- For $v \in V(G)$, $i \in \{1,2\}$, and $j \in \{1,2,\ldots,l_i(v)\}$, let $H_i^j(v)$ induces a graph which is isomorphic to H_i at v in $G \trianglerighteq_o H$.



EDGE-COMB PRODUCT

Lemma

Let W be a resolving set of $G \trianglerighteq_o H$.
Then $W \cap H_i^j(v) \neq \emptyset$.



For $o_i \in V(H)$, we say that a graph H_i is of:

- type $\mathcal{A}_{o_i}$ if there exists there exists a basis of H_i containing o_i.
- type $\mathcal{B}_{o_i}$ if H_i is not of type $\mathcal{A}_{o_i}$.

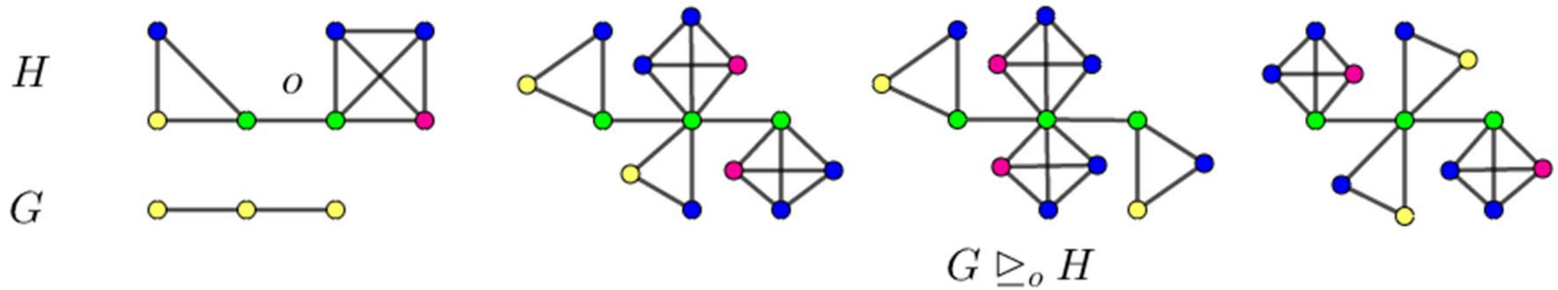
Lemma

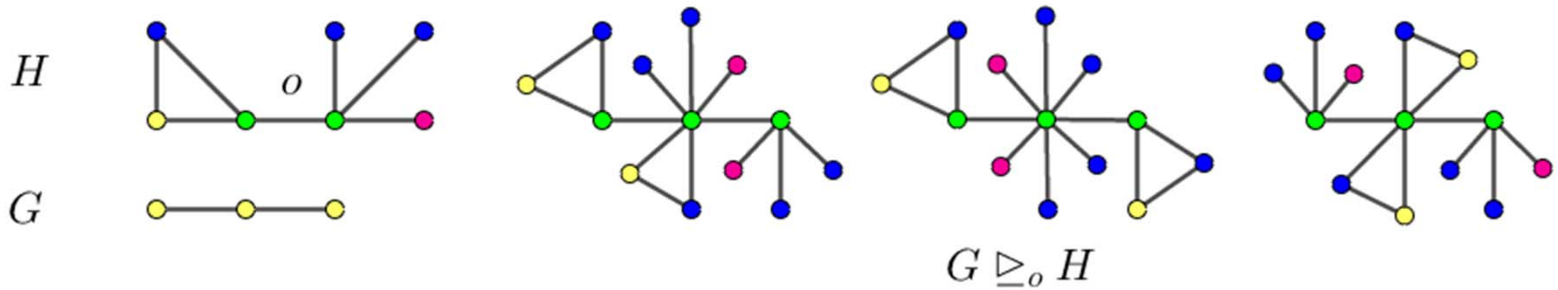
Let W be a resolving set of $G \trianglerighteq_o H$. Then

1. $|W \cap H_i^j(v)| \geq \beta(H_i) - 1$ if H_i is of type $\mathcal{A}_{o_i}$.
2. $|W \cap H_i^j(v)| \geq \beta(H_i)$ if H_i is of type $\mathcal{B}_{o_i}$.

EDGE-COMB PRODUCT

- type $\mathcal{A}_{o_i}$ if there exists there exists a basis of H_i containing o_i.
- type $\mathcal{B}_{o_i}$ if H_i is not of type $\mathcal{A}_{o_i}$.





EDGE-COMB PRODUCT

Let $o_i, z_i \in V(H_i)$ where z_i is adjacent to o_i and z_i is not an element of a basis of H_i. We say that a graph H_i is of:

- type $\mathcal{A}1_{o_i}$ if there exists there exists a basis W of H_i containing o_i such that there is a vertex $w \in W$ satisfying $d_{H_i}(w, z_i) \leq d_{H_i}(w, o_i)$.
- type $\mathcal{A}2_{o_i}$ if there exists there exists a basis W of H_i containing o_i such that every $w \in W$ satisfies $d_{H_i}(w, z_i) = d_{H_i}(w, o_i) + 1$.
- type $\mathcal{B}1_{o_i}$ if every basis W of H_i does not contain o_i and there is a vertex $w \in W$ satisfying $d_{H_i}(w, z_i) \leq d_{H_i}(w, o_i)$.
- type $\mathcal{B}2_{o_i}$ if every basis W of H_i does not contain o_i and every $w \in W$ satisfies $d_{H_i}(w, z_i) = d_{H_i}(w, o_i) + 1$.

For $X, Y \in \{\mathcal{A}1_{o_i}, \mathcal{A}2_{o_i}, \mathcal{B}1_{o_i}, \mathcal{B}2_{o_i}\}$, we say that H is of type (X, Y) if H_1 is of type X and H_2 is of type Y.

EDGE-COMB PRODUCT

Theorem

Let G and H be connected graphs of order at least 2. Let o be a bridge in H and $H \setminus \{o\} = H_1 \cup H_2$ where $H_1 \neq H_2$ and H_i $(1 \leq i \leq 2)$ is not a path. Then $\beta(G \trianglerighteq_o H) = |E(G)| \cdot (\beta(H_1) + \beta(H_2) - 2) + c$ where:

(i) $c = 0$ if H is of type $(\mathcal{A}1, \mathcal{A}1)$

(ii) $c = |E(G)|$ if H is of type $(\mathcal{A}1, \mathcal{B}1)$ or $(\mathcal{B}1, \mathcal{A}1)$

(iii) $c = 2 \cdot |E(G)|$ if H is of type $(\mathcal{B}1, \mathcal{B}1)$

(iv) $c = \sum_{v \in \Gamma_1(1)} (l_1(v) - 1)$ if H is of type $(\mathcal{A}2, \mathcal{A}1)$

(v) $c = \sum_{v \in \Gamma_1(2)} (l_2(v) - 1)$ if H is of type $(\mathcal{A}1, \mathcal{A}2)$

(vi) $c = |E(G)| + \sum_{v \in \Gamma_1(1)} (l_1(v) - 1)$ if H is of type $(\mathcal{B}2, \mathcal{A}1)$ or $(\mathcal{A}2, \mathcal{B}1)$

(vii) $c = |E(G)| + \sum_{v \in \Gamma_1(2)} (l_2(v) - 1)$ if H is of type $(\mathcal{A}1, \mathcal{B}2)$ or $(\mathcal{B}1, \mathcal{A}2)$

(viii) $c = 2 \cdot |E(G)| + \sum_{v \in \Gamma_1(1)} (l_1(v) - 1)$ if H is of type $(\mathcal{B}2, \mathcal{B}1)$

(ix) $c = 2 \cdot |E(G)| + \sum_{v \in \Gamma_1(2)} (l_2(v) - 1)$ if H is of type $(\mathcal{B}1, \mathcal{B}2)$

(x) $c = \sum_{v \in \Gamma_1(1)} (l_1(v) - 1) + \sum_{v \in \Gamma_1(2)} (l_2(v) - 1) + \max\{0, |\Gamma_{12}|\}$ if H is of type $(\mathcal{A}2, \mathcal{A}2)$

(xi) $c = |E(G)| + \sum_{v \in \Gamma_1(1)} (l_1(v) - 1) + \sum_{v \in \Gamma_1(2)} (l_2(v) - 1) + \max\{0, |\Gamma_{12}|\}$ if H is of type $(\mathcal{A}2, \mathcal{B}2)$ or $(\mathcal{B}2, \mathcal{A}2)$

(xii) $c = 2 \cdot |E(G)| + \sum_{v \in \Gamma_1(1)} (l_1(v) - 1) + \sum_{v \in \Gamma_1(2)} (l_2(v) - 1) + \max\{0, |\Gamma_{12}|\}$ if H is of type $(\mathcal{B}2, \mathcal{B}2)$

THANK YOU!

TERIMA KASIH