Classification of regular maps of genus 2

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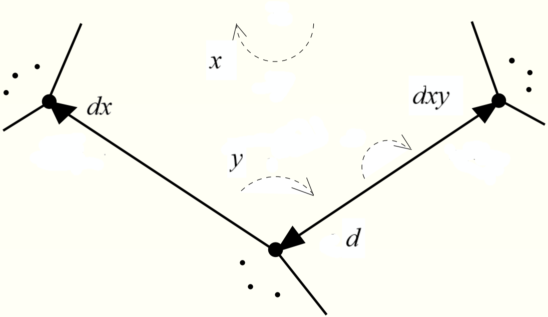
An (orientably-)-regular map is a 2-cell embedding of a graph on an orientable surface such that the group of all the orientation-preserving automorphisms of the embedding is transitive on arcs. Constant face boundary walk length (m) and vertex valency (n): M of type (m, n).



$$\mathrm{Aut}^+(M) = \langle x, y;\ x^m = y^n = (xy)^2 = \ldots = 1 \rangle$$

Regularity $\Rightarrow$ M can be identified with $\mathrm{Aut}^+(M)$ presented as above.

$G_i = \langle x_i, y_i;\ x_i^m = y_i^n = (x_i y_i)^2 = \ldots = 1\rangle$, $i = 1, 2$, represent isomorphic regular maps iff there is an isomorphism $G_1 \to G_2$: $(x_1, y_1) \mapsto (x_2, y_2)$.

If an (m, n)-group $G = \langle x, y;\ x^m = y^n = (xy)^2 = \ldots = 1\rangle$ represents a regular map of type (m, n), then $v = |G|/n$, $e = |G|/2$ and $f = |G|/m$. Moreover, if the carrier surface has genus g, by Euler's formula $(g \neq 1)$:

$$|G| = \frac{4}{1 - 2m^{-1} - 2n^{-1}}(g - 1) = \nu(m, n)(g - 1)$$

Thus, if $g \geq 2$, then $1/m + 1/n < 1/2$ and as for such hyperbolic pairs (m, n) one has $1/m + 1/n \leq 1/3 + 1/7$, $\Rightarrow |G| \leq 84(g - 1)$ for $g \geq 2$.

Classification of regular maps of genus 2 boils down to classification of (m, n)-groups of genus 2, i.e., those for which $|G| = \nu(m, n)$ $(m \geq n)$.

Let us begin by compiling a list of positive integer values of $\nu(m, n)$:

m	7	8	9	10	12	18	5	6	8	12	5	10	6	8
n	3	3	3	3	3	3	4	4	4	4	5	5	6	8
$\|G\|$	84	48	36	30	24	18	40	24	16	12	20	10	12	8

- *There are no (m,n)-groups of genus 2 in the following 8 cases:*

m	7	9	10	12	18	5	12	5
n	3	3	3	3	3	4	4	5
$\lvert G\rvert$	84	36	30	24	18	40	12	20

- (7,3): consider implications of Sylow theorems applied to the Sylow 7-subgroups of such a group of order 84.

- (5,4): an analogous argument works for the Sylow 5-subgroups.

- (9,3): cannot have $G' = G$ and so, abelianising the presentation $G = \langle x, y;\ x^9 = y^3 = (xy)^2 = \ldots = 1\rangle$ one arrives at $G/G' \cong C_3$ and hence $|G'| = 12$, so that G' is one of the 5 groups of order 12. Considering characteristic subgroups H of order 4 if $G' = A_4$ and of order 3 in the remaining four cases for G', derive a contradiction by looking at the presentation of G/H in terms of $\overline{x} = xH$ and $\overline{y} = yH$.

- The remaining items are much easier to handle (in a similar way). □

• *There are: a unique* $(8,8)$*-group of order* 8*, a unique* $(10,5)$*-group of order* 10*, a unique* $(6,6)$*-group of order* 12*, and a unique* $(8,4)$*-group of order* 16*, with presentations*

$$\begin{array}{ll}\langle x,y;\ x^8=y^8=(xy)^2=x^3y^{-1}=1\rangle & \cong C_8\\ \langle x,y;\ x^{10}=y^5=(xy)^2=x^4y^{-1}=1\rangle & \cong C_{10}\\ \langle x,y;\ x^6=y^6=(xy)^2=[x,y]=1\rangle & \cong C_6\times C_2\\ \langle x,y;\ x^8=y^4=(xy)^2=yxy^{-1}x^{-3}=1\rangle & \cong C_8\rtimes C_2\end{array}$$

• The case $(8,4)$. Letting $z=xy$, from $\langle x\rangle \triangleleft G=\langle x,y\rangle$ of order 16 one has $zxz=x^{\varepsilon}$ for $\varepsilon\in\{\pm1,\pm3\}$. If $\varepsilon=+1$, then $G\cong C_8\times C_2$, and for $\varepsilon=-1$ one obtains $G\cong D_{16}$, giving quick contradictions in both cases.

If $\varepsilon=-3$, i.e., $zxz=x^{-3}$, then $x^{-1}zx^{-1}z=x^2$, but $x^{-1}z=y$, which implies that x^2 and y^2 have the same order, a contradiction.

The only survivor is $\varepsilon=3$, i.e., $zxz=x^3$, implying $x^{-1}zx^{-1}z=x^4$ and giving y of order 4, together with the existence of the group as claimed. □

- *Unique* $(6,4)$*-group* $\langle x, y;\ x^6 = y^4 = (xy)^2 = (x^{-1}y)^2 = 1\rangle \cong C_3 \rtimes D_8$.

- Take $G = \langle x, y \mid x^6 = y^4 = (xy)^2 = \ldots = 1\rangle$, $|G| = 24$. If $\langle x\rangle \cap \langle y\rangle \neq 1$, then $L = \langle x^3\rangle = \langle y^2\rangle \triangleleft G$ and inspecting the presentation of G/L gives $|G/L| \leq 6$, a contradiction. It follows that $\langle x\rangle \cap \langle y\rangle = 1$, and $G = \langle x\rangle\langle y\rangle$.

Let $H = \langle x\rangle$; let $\theta :\ G \to S_4$ be perm-rep induced by left multiplication by elements of G on the set $G/H = \{y^i H;\ 0 \leq i \leq 3\}$ of left cosets of H. As S_4 has no elts of order 6, $\mathrm{Im}(\theta)$ is a proper subgroup of S_4. Further, as $\theta(y)$ is a 4-cycle, we cannot have $\theta(G) \cong A_4$ and so $|\theta(G)| \leq 8$.

Now, $(xy)^2 = 1$ implies $x \cdot yH = y^3 H$ and so $\theta(x)$ has order 2 in $\mathrm{Im}(\theta)$. This means that $\mathrm{Im}(\theta) \cong D_8$, and so $x \cdot y^3 H = yH$ and $x \cdot y^2 H = y^2 H$, with both equations reducing to $yx^{-1}y \in \{x, x^{-1}\}$. But if $yx^{-1}y = x^{-1}$, then $\langle y\rangle \triangleleft G$ and by our presentation of G we would have $G/\langle y\rangle$ of order at most 2, a contradiction. Thus, $yx^{-1}y = x$, that is, $(x^{-1}y)^2 = 1$, $\Rightarrow$

$$G \textit{ is a quotient of } \langle x, y;\ x^6 = y^4 = (xy)^2 = (x^{-1}y)^2 = 1\rangle\ .$$

The above relations imply $x^3 y x^3 = y^{-1}$, so that $\langle x^3, y\rangle \cong D_8$ and hence

$$G = \langle x^2, y\rangle\langle x^3\rangle = \langle x^2\rangle \rtimes \langle x^3, y\rangle \cong C_3 \rtimes D_8\ . \quad \square$$

- *There is a unique $(8,3)$-group of order 48.*

- Let $G = \langle x, y;\ x^8 = y^3 = (xy)^2 = \ldots = 1\rangle$, $|G| = 48$. For $H = \langle x \rangle$, consider again a perm-rep $\theta:\ G \to S_6$ given by left multiplication by elements of G on the set of 6 left cosets G/H, containing H and yH.

As S_6 does not contain an element of order 8, the image of x^4 under θ must be the identity permutation of G/H. It follows that $x^4 \cdot yH = yH$, that is, $y^{-1}x^4y$, an element of order 2, must lie in $H = \langle x \rangle$. But the only involution in H is x^4, and so $[x^4, y] = 1$. We thus conclude that:

$$G \text{ is a quotient of } \langle x, y;\ x^8 = y^3 = (xy)^2 = [x^4, y] = 1 \rangle\ .$$

As the expected order 48 has a suspicious form $(p^2-1)(p^2-p)$ for $p = 3$, one is tempted to try $G = \mathrm{GL}(2,3)$ ... and, indeed, the assignment

$$x \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

defines an isomorphism of our $(8,3)$-group onto $G = \mathrm{GL}(2,3)$. $\square$

• *Up to isomorphism and duality there are exactly* 6 *orientably-regular maps* M *of genus* 2*, with* $\mathrm{Aut}^+(M) = \langle x, y\rangle$ *presented as displayed:*

$$\begin{array}{ll}
\langle x, y;\ x^8 = y^8 = (xy)^2 = x^3y^{-1} = 1\rangle & \cong C_8 \\
\langle x, y;\ x^{10} = y^5 = (xy)^2 = x^4y^{-1} = 1\rangle & \cong C_{10} \\
\langle x, y;\ x^6 = y^6 = (xy)^2 = [x, y] = 1\rangle & \cong C_6 \times C_2 \\
\langle x, y;\ x^8 = y^4 = (xy)^2 = yxy^{-1}x^{-3} = 1\rangle & \cong C_8 \rtimes C_2 \\
\langle x, y;\ x^6 = y^4 = (xy)^2 = (xy^{-1})^2 = 1\rangle & \cong C_3 \rtimes D_4 \\
\langle x, y;\ x^8 = y^3 = (xy)^2 = (x^2y)^3 = [x^4, y] = 1\rangle & \cong \mathrm{GL}(2, 3)
\end{array}$$

A regular map $\leftrightarrow G = \langle x, y;\ x^m = y^n = (xy)^2 = \ldots = 1\rangle$ is reflexible iff G has an automorphism fixing xy and inverting $y \Leftrightarrow$ inverting both x, y.

• *All the six regular maps of genus* 2 *from above are reflexible.*

• E.g. take $R(x, y) = yxy^{-1}x^{-3}$ of the $(8, 4)$-group. Replacing x, y with inverses gives $R(x^{-1}, y^{-1}) = y^{-1}x^{-1}yx^3$. The relation $R(x^{-1}, y^{-1}) = 1$ is equivalent to $yx^3y^{-1} = x$; cubing both sides $\Rightarrow yxy^{-1} = x^3 \Leftrightarrow R(x, y)$.$\Box$

Thank you.