The Erdős–Rothschild problem

joint work with Oleg Pikhurko (Warwick)

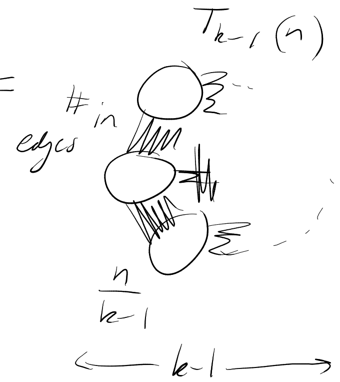
Turán's theorem 1941: What is the maximum number of edges in an n-vertex graph that is K_k-free? $ex(n, K_k)$

$\triangle = K_3$

Erdős–Rothschild problem 1974: What is the maximum number of ways to partition (colour) the edges of an n-vertex graph into s pieces where each piece is K_k-free?

$F_s(n,k) = \max_{v(G)=n} F_s(G,k)$

Turán $ex(n, K_k) =$ # edges in



Turán graph unique

$n=6$, $s=2$, $k=3$

$F_2(K_6, \triangle) = 0$ Ramsey: $R(3)=6$

$F_2(K_{3,3}, \triangle) = 2^{3\cdot 3}$

$\rightsquigarrow$ Lower bound $F_s(n,k) \geq F_s(T_{k-1}(n), k) = s^{ex(n,K_k)}$

Conjecture (ER, 1974) Tight for $s=2$ (col), $k=3$ (clique)

Theorem (Yuster '96, Alon–Balogh–Keevash–Sudakov '04)

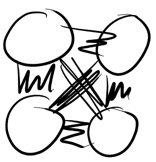
YES for all $n \geq 6$

$s = 2,3$ YES for large n all k $F_s(n,k) = F_s(T_{k-1}(n), k)$ unique

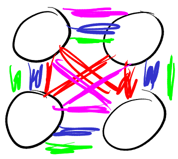
$s \geq 4$ NO

Thm (Pikhurko–Yilma '10) $F_4(n, \triangle) = F_4(T_4(n), \triangle)$ unique

$F_4(n, \boxtimes) = F_4(T_9(n), \boxtimes)$



99% of colourings look like

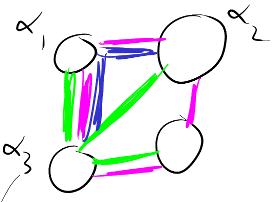


Thm (Botler–Corsten–Dankovics–Frankl–Hàn–Jiménez–Skokan '21+)

$F_5(n, \triangle)$ 'continuous' family of almost extremal graphs

$F_6(n, \triangle) = F_6(T_8(n), \triangle)$ unique.

An optimisation problem



- $(\alpha_1, \ldots, \alpha_r)$ $\alpha_i \geq 0$ $\sum a_i = 1$
- colour pattern $\phi : \{ij : 1 \leq i < j \leq r\} \to$ subsets of [5]

$\phi^{-1}(c)$ is K_h-free

$h = 3$

$F_5(n, h) \geq$ # of colourings of a blow-up of $(\alpha_1, \ldots, \alpha_r)$ that come from ϕ.

$$3^{\alpha_1 n \cdot \alpha_3 n} \cdot 2^{\alpha_1 n \cdot \alpha_2 n} \cdots$$

$$F_5(n, h) \geq \prod_{ij} |\phi(ij)|^{\alpha_i n \, \alpha_j n}$$

$$\frac{\log_2 F_5(n,h)}{n^2/2} \geq 2 \sum_{ij} \alpha_i \alpha_j \log |\phi(ij)| = q(r, \phi, \underline{\alpha})$$

MAXIMISE

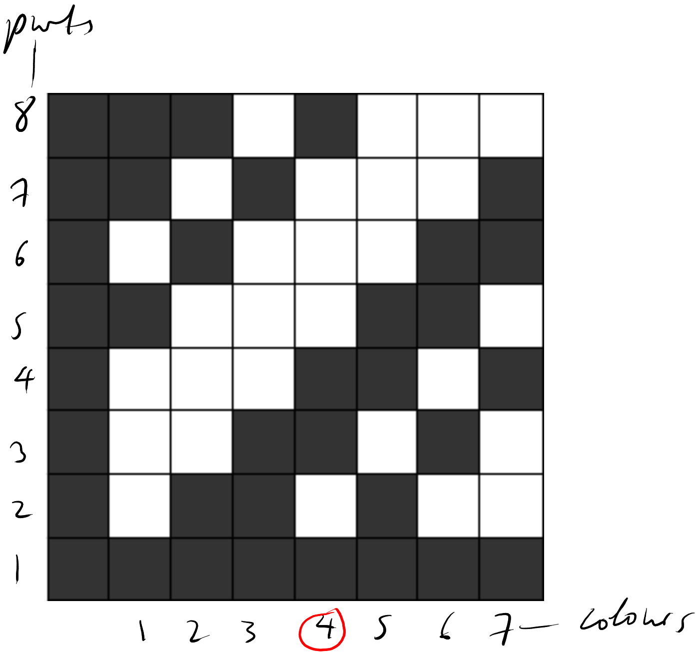
Theorem (Pikhurko–Yilma–S '16, Pikhurko–S, '21$^+ \times 2$)

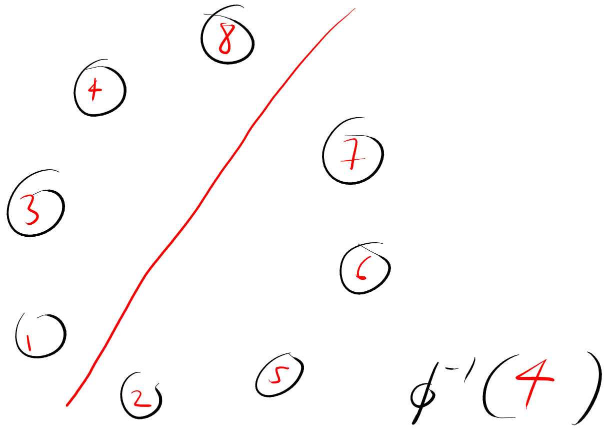
To determine $F_5(n,h)$ exactly it suffices to solve OPT problem.

Given solutions to OPT problem, we can often solve ER problem.

- OPT is a finite problem. e.g. $r \le R_s(k)$ Ramsey number
- Recover all previous results
- New application $F_7(n, \Delta) = F_7(T_8(n), \Delta)$ (Pikhurko-S)

unique





99% of colourings of $T_8(n)$ come from this chessboard

Hadamard matrix ■ = 1 $\quad H_n H_n^T = n I_n$

H_8 □ = −1

- Hadamard matrices have max determinant among all $n \times n$ matrices with entries in the complex unit circle.
- H_{4t} exists $\iff$ $(4t-1)K_{4t}$ has a $K_{2t,2t}$-decomposition
- All known Δ results come from Hadamard matrices

  True in general?

- Do Hadamard matrices H_{4t} exist?