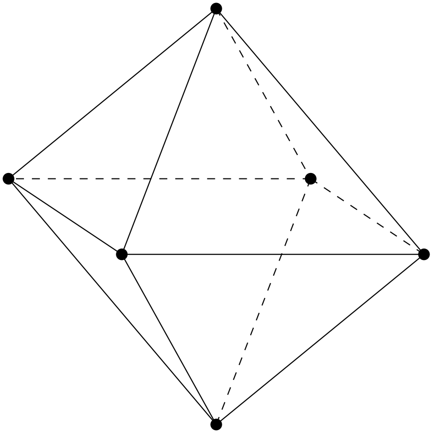
Spectra of Farey Maps

Margaret Stanier

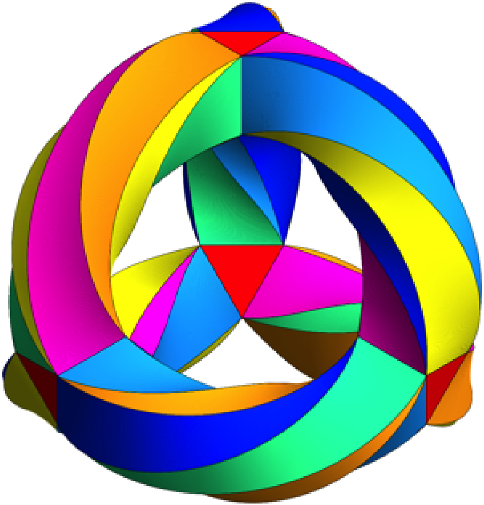
2nd February 2022

Outline

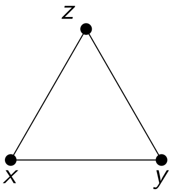
The Farey map of level 4



The Farey map of level 7

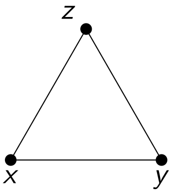


The spectra of maps and graphs



- The underlying graph of the Farey map of level 2 is the complete graph on 3 vertices, K_3.

The spectra of maps and graphs



- The underlying graph of the Farey map of level 2 is the complete graph on 3 vertices, K_3.
- Its adjacency matrix is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The spectra of maps and graphs

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

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- The eigenvalues are $2, -1, -1$.
- The spectrum is $\mathrm{sp}(K_3) = \{(-1)^{(2)}, 2\}$.

Example

Consider the following matrices, whose elements are in $\mathbb{Z}/2\mathbb{Z}$:

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{generate}$$

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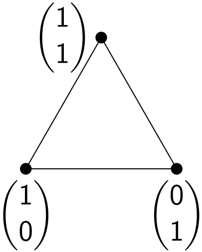


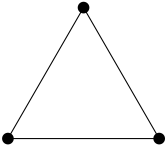
Regular maps on oriented surfaces

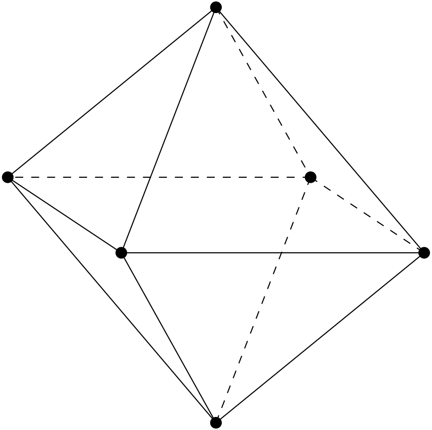
G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, Proc. London Math. Soc. (3) **37** (1978), no. 2, 273–307.

K. Hu, G.A. Jones, R. Nedela and N.-E. Wang, Non-abelian almost totally branched coverings over the platonic maps, European J. Combin. **51** (2016), 1–11.

- If x is an element of order 2 in a group G, and the elements x and y generate G, then $\mathcal{M} = (G, x, y)$ is a regular map on an oriented surface.
- The darts of $\mathcal{M}$ are identified with the elements of G.
- The edges of $\mathcal{M}$ are identified with the left cosets of x in G.
- The vertices of $\mathcal{M}$ are identified with the left cosets of y in G.
- The faces of $\mathcal{M}$ are identified with the left cosets of xy^{-1} in G.

Example of a map covering





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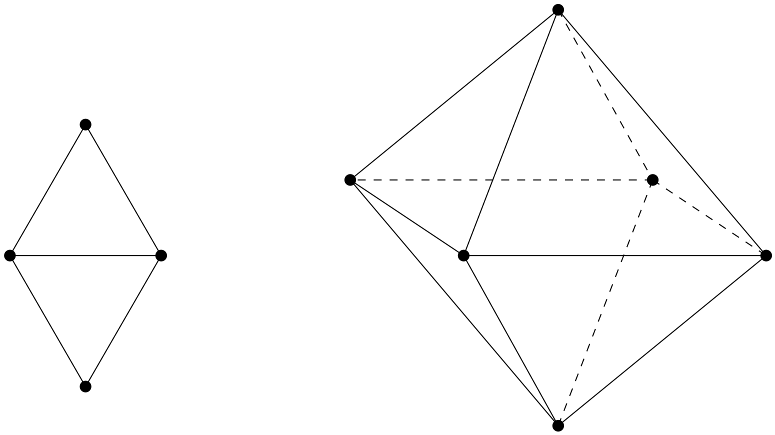


Map coverings

Let $\mathcal{M}_1 = (G_1, x_1, y_1)$ and $\mathcal{M}_2 = (G_2, x_2, y_2)$ be two regular maps on oriented surfaces. y_1 is of order n_1, and y_2 is of order n_2. Then

- If there is a group homomorphism σ from G_1 to G_2, then σ is a covering transformation and $\mathcal{M}_1$ is a covering of $\mathcal{M}_2$.
- If $\mathcal{M}_1$ is a covering of $\mathcal{M}_2$, then n_2 divides n_1. Put $n_1 = dn_2$.
- The covering is ramified at the vertices with ramification index d.

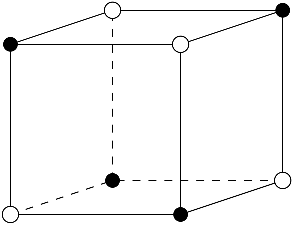
Lemma

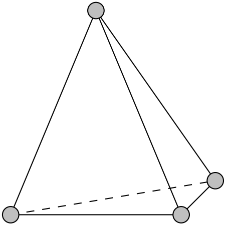
If v and v' are adjacent vertices of $\mathcal{M}_2$, and w is in the fibre of v, then it is adjacent to exactly d vertices in the fibre of v'.

Example of a graph covering

○ even vertex

● odd vertex





Graph coverings

- A graph covering transformation from a graph $\mathcal{G}_1$ to a graph $\mathcal{G}_2$ is a mapping between vertex sets such that there is a bijection between the set of vertices adjacent to a vertex of $\mathcal{G}_1$ and the set of vertices adjacent to any vertex of $\mathcal{G}_2$.
- If $\mathcal{G}_1$ is a graph covering of $\mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same vertex valency.
- If $\mathcal{G}_1$ is a graph covering of $\mathcal{G}_2$, and v and v' are two adjacent vertices of $\mathcal{G}_2$, then any vertex of $\mathcal{G}_1$ which is in the fibre of v is adjacent to exactly one vertex in the fibre of v'.
- If $\mathcal{G}_1$ and $\mathcal{G}_2$ are the underlying graphs of two maps, the graph covering may not be a map covering.

Spectrum of a covering map or graph

Lemma

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two regular graphs with vertex valencies n and m respectively, where $n = dm$, and suppose either that they are the underlying graphs of two regular maps $\mathcal{M}_1$ and $\mathcal{M}_2$, and that $\mathcal{M}_1$ is a regular map covering of $\mathcal{M}_2$, or, if $d = 1$, that $\mathcal{G}_1$ is a graph covering of $\mathcal{G}_2$. Let N be the number of vertices of $\mathcal{G}_2$. Then

1. $d\,\mathrm{sp}(\mathcal{G}_2) \subset \mathrm{sp}(\mathcal{G}_1)$, *and*
2. *if* $d > 1$, $d\,\mathrm{sp}(\mathcal{G}_2) \cup \{0^{(\gamma)}\} \subset \mathrm{sp}(\mathcal{G}_1)$, *where* $\gamma \geq N(d-1)$.

Sketch of a proof of part 1

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ a \\ b \\ c \end{pmatrix} = d\lambda \begin{pmatrix} a \\ b \\ c \\ a \\ b \\ c \end{pmatrix}.$$

Sketch of a proof of part 2

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ -a \\ -b \\ -c \end{pmatrix} = 0 \begin{pmatrix} a \\ b \\ c \\ -a \\ -b \\ -c \end{pmatrix}.$$

Lemma

Suppose that A is an $N \times N$ symmetric matrix, and that there is a positive integer n such that

$$A^2 - nI \quad = \quad \begin{pmatrix} C & C & \cdots & C \\ C & C & \cdots & C \\ \vdots & \vdots & \ddots & \vdots \\ C & C & \cdots & C \end{pmatrix},$$

where C is an $r \times r$ symmetric matrix. Then either $\sqrt{n}$ or $-\sqrt{n}$ or both are eigenvalues of A with total algebraic multiplicity greater than or equal to $N - \operatorname{rank}(C)$.

The modular group

The modular group is

$$\Gamma = \mathrm{PSL}_2(\mathbb{Z}) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

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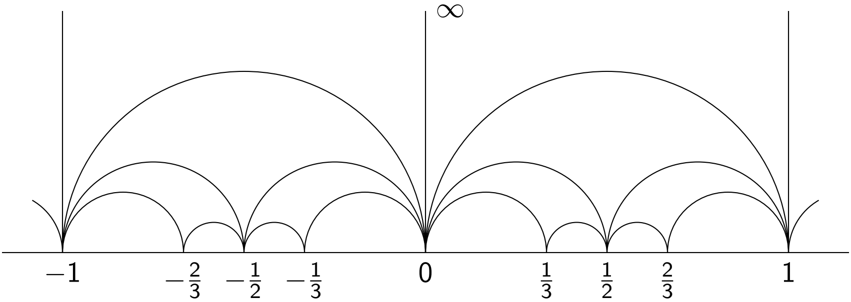
$$\Gamma = \mathrm{PSL}_2(\mathbb{Z}) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Its principal congruence groups are, for all positive integers n,

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}.$$

The Farey tessellation



Part of the Farey tessellation $\mathcal{F}$ drawn on the extended hyperbolic plane $\hat{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. It is an infinite triangulation of this plane.

- Vertices are the set $\mathbb{Q} \cup \infty$.
- Two vertices $\frac{a}{c}$ and $\frac{b}{d}$ are adjacent if $ad - bc = \pm 1$.
- The neighbours of infinity are integers.
- The edges joining adjacent vertices are hyperbolic geodesics: semi-circles or vertical lines.

Maps are embeddings of graphs on surfaces

D. Singerman, Universal tessellations, Rev. Mat. Univ. Complut. Madrid **1** (1988), no. 1-3, 111–123.

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Theorem: Singerman 1988

Any subgroup K of $\mathrm{PSL}_2(\mathbb{Z})$ gives rise to a triangular map $\mathcal{F}/K$ on the surface $\mathbb{H}/K$.

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Any normal subgroup K of $\mathrm{PSL}_2(\mathbb{Z})$ gives rise to a regular triangular map $\mathcal{F}/K$ on the surface $\mathbb{H}/K$.

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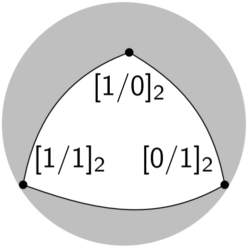
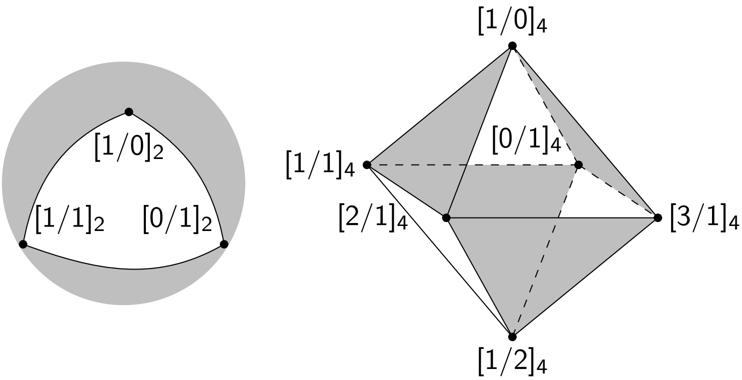
The Farey map of level n is $\mathcal{M}_3(n) = \mathcal{F}/\Gamma(n)$.

The Farey maps of level 2 and 4



$\mathcal{M}_3(4)$ is a regular cover of order 4 of $\mathcal{M}_3(2)$, ramified at the vertices.

The Farey maps of level 2 and 4



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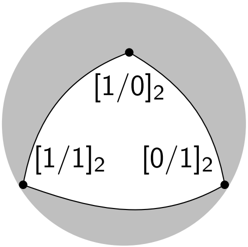

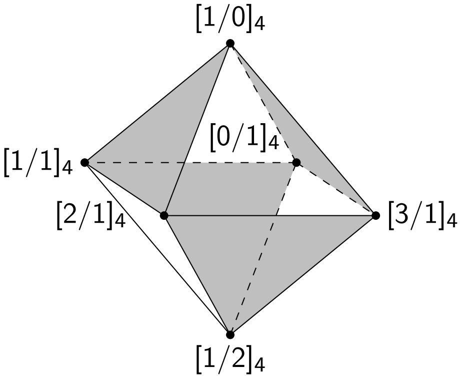

$\mathcal{M}_3(4)$ is a regular cover of order 4 of $\mathcal{M}_3(2)$, ramified at the vertices.

What is the relationship between these maps?

- For a prime p and a positive integer k, if $p^k > 2$, $\mathcal{M}_3(p^{k+1})$ is a regular covering of $\mathcal{M}_3(p^k)$ of order p^3 ramified at the vertices with ramification index p.

Parallel products of maps

- The *tensor product* or *direct product* of two graphs: $\mathcal{G}_1$ with dart set Ω_1, and $\mathcal{G}_2$ with dart set Ω_2 is the graph whose dart set is the cartesian product $\Omega_1 \times \Omega_2$.
- If r and s are coprime integers, the tensor product of the underlying graphs $\mathcal{G}(r)$ and $\mathcal{G}(s)$ of $\mathcal{M}_3(r)$ and $\mathcal{M}_3(s)$ is the underlying graph of a map - the parallel product of $\mathcal{M}_3(r)$ and $\mathcal{M}_3(s)$.

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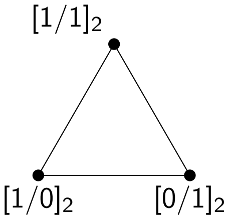
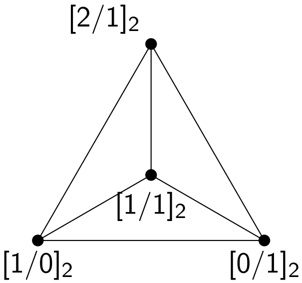
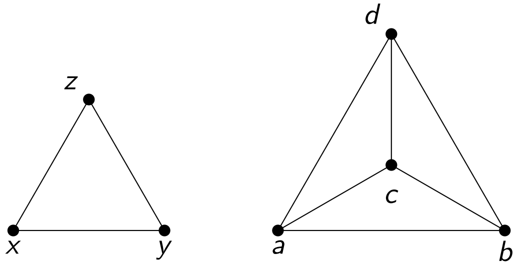
Parallel products of maps





Parallel products of maps



- The vertices of the parallel product of the maps $\mathcal{M}_3(2)$ and $\mathcal{M}_3(3)$ are xa, xb, xc, xd; ya, yb, yc, yd; za, zb, zc and zd.
- The vertex xa is adjacent to the vertices yb, yc, yd, zb, zc and zd.

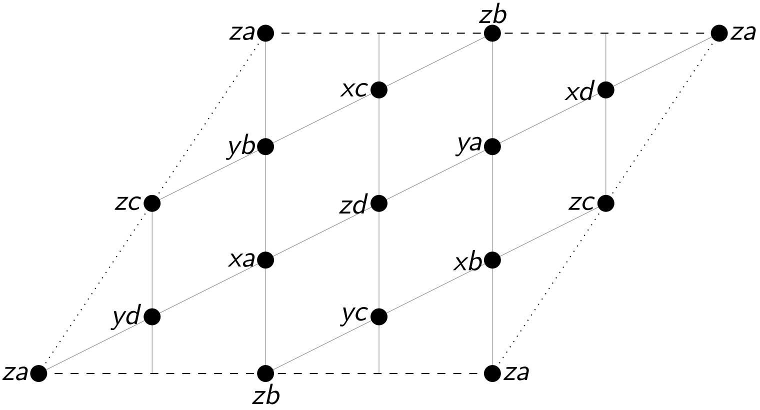


Figure: The map of the parallel product of $\mathcal{M}_3(2)$ and $\mathcal{M}_3(3)$, drawn on a torus.

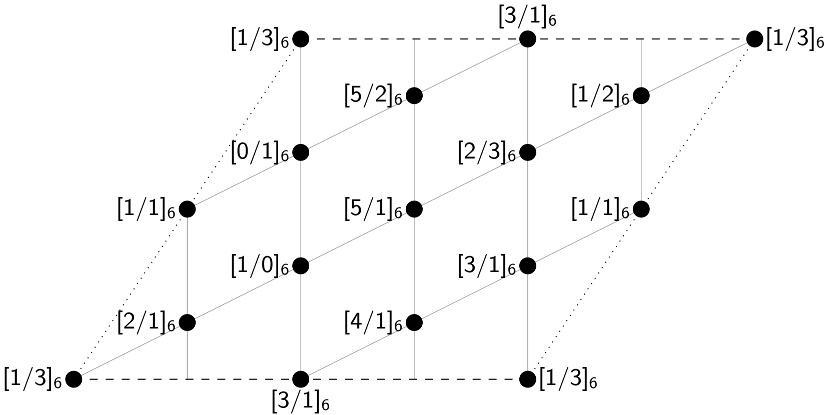


Figure: The map $\mathcal{M}_3(6)$, drawn on a torus.

What is the relationship between these maps?

1. For a prime p and a positive integer k, with the exception of the case $p = 2,\ k = 1$, $\mathcal{M}_3(p^{k+1})$ is a regular covering of $\mathcal{M}_3(p^k)$ of level p^3 ramified at the vertices with ramification index p.
2. For an odd integer m, $\mathcal{M}_3(2m)$ is the parallel product of the maps $\mathcal{M}_3(2)$ and $\mathcal{M}_3(m)$.

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3. For coprime positive integers $l, m > 2$, neither of which is twice an odd integer, $\mathcal{M}_3(lm)$ is a regular unramified double covering of the parallel product of $\mathcal{M}_3(l)$ and $\mathcal{M}_3(m)$.

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4. For a prime $p > 2$, $\mathcal{G}(p)$ is a regular graph covering of the complete graph on $p + 1$ vertices.

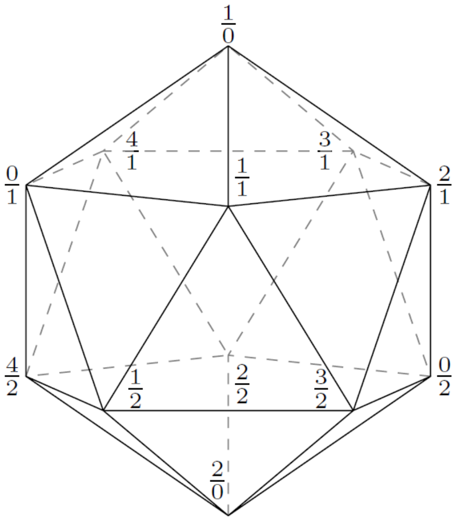
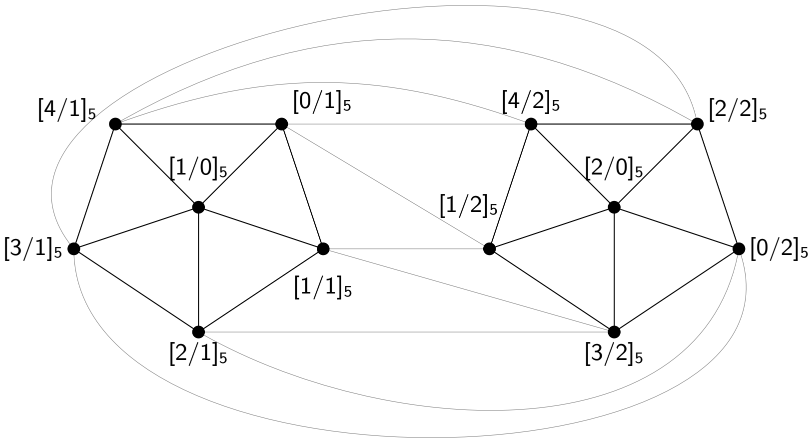


Figure: The map $\mathcal{M}_3(5)$, an icosahedron embedded in a sphere.


[4/1]$_5$
[0/1]$_5$
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[2/1]$_5$
[3/2]$_5$

The adjacency matrix of the Farey map of level 5

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The square of the adjacency matrix of the Farey map of level 5

$$A^2 = \begin{pmatrix} 5 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 5 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 5 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 5 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 5 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 5 & 2 & 2 & 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 2 & 2 & 2 & 5 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 & 2 & 5 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 5 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 5 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 5 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 5 \end{pmatrix}.$$

The square of the adjacency matrix of the Farey map of level 5

$$A^2 - 5I = \left(\begin{array}{cccccc|cccccc} 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 \\ \hline 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 0 \end{array}\right).$$

Spectra of Farey maps of prime level

- $\mathrm{sp}(K(n)) = \{(-1)^{(n-1)}, n-1\}$.
- $\mathrm{sp}(\mathcal{M}_3(2)) = \{(-1)^{(2)}, 2\}$; $\mathrm{sp}(\mathcal{M}_3(3)) = \{(-1)^{(3)}, 3\}$.

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- $\mathrm{sp}(\mathcal{M}_3(2)) = \{(-1)^{(2)}, 2\}$; $\mathrm{sp}(\mathcal{M}_3(3)) = \{(-1)^{(3)}, 3\}$.

$\mathcal{M}_3(5)$ is a graph covering of $K(6)$, so its spectrum includes $\{(-1)^{(5)}, 5\}$. $A^2(5) - 5I$ comprises 2×2 blocks which are each of rank 6, so the spectrum includes $\pm\sqrt{5}$ with total multiplicity $12 - 6 = 6$. The eigenvalues of an adjacency matrix sum to 0 as its trace is 0, so $\sqrt{5}$ and $-\sqrt{5}$ have the same multiplicity, and

$$\mathrm{sp}(\mathcal{M}_3(5)) = \{(-\sqrt{5})^{(3)}, (-1)^{(5)}, (\sqrt{5})^{(3)}, 5\}.$$

Spectra of Farey maps of prime level

- $\text{sp}(K(n)) = \{(-1)^{(n-1)}, n-1\}$.
- $\text{sp}(\mathcal{M}_3(2)) = \{(-1)^{(2)}, 2\}$; $\text{sp}(\mathcal{M}_3(3)) = \{(-1)^{(3)}, 3\}$.

$\mathcal{M}_3(5)$ is a graph covering of $K(6)$, so its spectrum includes $\{(-1)^{(5)}, 5\}$. $A^2(5) - 5I$ comprises 2×2 blocks which are each of rank 6, so the spectrum includes $\pm\sqrt{5}$ with total multiplicity $12 - 6 = 6$. The eigenvalues of an adjacency matrix sum to 0 as its trace is 0, so $\sqrt{5}$ and $-\sqrt{5}$ have the same multiplicity, and

$$\text{sp}(\mathcal{M}_3(5)) = \{(-\sqrt{5})^{(3)}, (-1)^{(5)}, (\sqrt{5})^{(3)}, 5\}.$$

Theorem

For a prime p, $\text{sp}(\mathcal{M}_3(p)) = \{(-\sqrt{p})^{(m)}, (-1)^{(p)}, (\sqrt{p})^{(m)}, p\}$, where $m = \frac{1}{4}(p-3)(p+1)$

Spectra of Farey maps of prime power level

- $\mathrm{sp}(\mathcal{M}_3(25))$ includes

$$\{(-5\sqrt{5})^{(3)}, (-5)^{(5)}, 0^{(48),}(5\sqrt{5})^{(3)}, 25\}.$$

Spectra of Farey maps of prime power level

- $\mathrm{sp}(\mathcal{M}_3(25))$ includes

$$\{(-5\sqrt{5})^{(3)}, (-5)^{(5)}, 0^{(48),}(5\sqrt{5})^{(3)}, 25\}.$$

- $\mathcal{M}_3(25)$ has $12 \times 25 = 300$ vertices. We can arrange its 300 rows and columns so that $A^2(25) - 25I$ consists of 5×5 copies of

$$B = \begin{pmatrix} C & D & D & D & D \\ D & C & D & D & D \\ D & D & C & D & D \\ D & D & D & C & D \\ D & D & D & D & C \end{pmatrix}, \text{ where } C = A^2(5) - 5I \text{ and } D = A^2(5).$$

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$$\mathrm{sp}\mathcal{M}_3(25)) = \{(-5\sqrt{5})^{(3)}, (-5)^{(125)}, 0^{(48)}, 5^{(120)}(5\sqrt{5})^{(3)}, 25\}.$$

Spectra of Farey maps of prime power level

Theorem

$sp(\mathcal{M}_3(4)) = \{(-1)^{(2)}, (0)^{(3)}, 4\}$. For $p^k \neq 4$, if $\mathcal{M}_3(p^{k-1})$ has N vertices and $\mathrm{sp}(\mathcal{M}_3(p^{k-1})) = \{\lambda_i^{(m_i)}, i = 1, 2, \dots, r\}$, then $\mathrm{sp}(\mathcal{M}_3)(p^k)) = \{(p\lambda_i)^{(m_i)} \cup \{(-\sqrt{p^k})^{(m)}, (0)^{((p-1)N)}, (\sqrt{p^k})^{(m)}\}$, where $m = \frac{1}{2}p(p-1)N$.

Spectra of Farey maps of composite level

- If λ is an eigenvalue of a graph $\mathcal{G}_1$ and μ is an eigenvalue of a graph $\mathcal{G}_2$, then $\lambda\mu$ is an eigenvalue of their direct product.
- $\mathcal{M}_3(6)$ is the parallel product of $\mathcal{M}_3(2)$ and $\mathcal{M}_3(2)$.
- So $\mathrm{sp}(\mathcal{M}_3(6))$ is the multiset

$$\{(-1)^{(2)}, 2\}\{(-1)^{(3)}, 3\} = \{(-3)^{(2)}, (-2)^{(3)}, 1^{(6)}, 6\}.$$

Theorem

Let m be an odd integer. If $\mathrm{sp}(\mathcal{M}_3(m)) = \{\lambda_i^{(m_i)}, i = 1, 2, \ldots, r\}$ then $\mathrm{sp}(\mathcal{M}_3(2m)) = \{(2\lambda_i)^{(m_i)}, (-\lambda_i)^{2m_i},\ i = 1, 2, \ldots, r\}$.

Spectra of Farey maps of composite level

- $\mathcal{M}_3(12)$ has 24 vertices. It is an unramified double covering of the parallel product of $\mathcal{M}_3(4)$ and $\mathcal{M}_3(3)$, which has 12 vertices.
- So $\mathrm{sp}(\mathcal{M}_3(12))$ contains the multiset $\{(-1)^{(2)}, 0^{(3)}, 4\}\{(-1)^{(3)}, 3\} = \{(-4)^{(3)}, (-3)^{(2)}, 0^{(12)}, 1^{(6)}, 12\}$.
- $A^2(12) - 12I$ can be written as the block matrix $\begin{pmatrix} B & B \\ B & B \end{pmatrix}$, where B is the adjacency matrix of the parallel product of $\mathcal{M}_3(4)$ and $\mathcal{M}_3(3)$. So $\pm 2\sqrt{3}$ are eigenvalues of $\mathcal{M}_3(12)$ with total multiplicity greater than or equal to $24 - 12 = 12$.
- Therefore $\mathrm{sp}(\mathcal{M}_3(12)) = \{(-4)^{(3)}, (-3)^{(2)}(-2\sqrt{3})^{(6)}, 0^{(12)}, 1^{(6)}, (2\sqrt{3})^{(6)}, 12\}$.

Spectra of Farey maps of composite level

Theorem

For coprime integers $l, m > 2$*, neither of which is twice an integer, if*
$\mathrm{sp}(\mathcal{M}_3(l)) = \{\lambda_i^{(m_i)}, i = 1, \ldots, r\}$*, and*
$\mathrm{sp}(\mathcal{M}_3(m)) = \{\mu_j^{(m_j)}, i = 1, \ldots, s\}$*, then*
$\mathrm{sp}(\mathcal{M}_3(lm)) = \{(\lambda_i \mu_j)^{(m_i m_j)}\} \cup \{(-\sqrt{lm})^{(N/4)}, (\sqrt{lm})^{(N/4)}\}$.

Hecke maps

- The Hecke group H^q is a discrete subgroup of infinite index in $\mathrm{PSL}_2(\mathbb{Z}[\lambda_q])$ generated by the matrices

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & \lambda_q \\ 0 & 1 \end{bmatrix}, \quad \text{where } \lambda_q = 2\cos \pi/q.$$

- The universal Hecke map $\widehat{\mathcal{M}}_q$ is the tessellation of the upper hyperbolic plane whose darts are the dart from infinity to zero and its images under H^q. If $q = 3$, then $\lambda_q = 1$, so H^3 is the modular group and $\widehat{\mathcal{M}}_3$ is the Farey tessellation.
- Hecke maps are quotients of $\widehat{\mathcal{M}}_q$ by the congruence subgroups of H^q.
- The maps $\mathcal{M}_4(n)$ and $\mathcal{M}_6(n)$ correspond to congruence subgroups of the Hecke groups H^4 and H^6. These are relatively straightforward to deal with as $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$. The faces of $\mathcal{M}_4(n)$ and $\mathcal{M}_6(n)$ are, respectively, quadrilaterals and hexagons.

Hecke maps

$\mathcal{M}_4(n)$ has two types of vertices.

- If (a, c) is an ordered pair in $\mathbb{Z} \times \mathbb{Z}$ such that $\gcd(a, c, n) = 1$ and $\gcd(a, 2, n) = 1$, we write an *even vertex* $[a/c\sqrt{2}]_n$.
- If (a, c) is an ordered pair in $\mathbb{Z} \times \mathbb{Z}$ such that $\gcd(a, c, n) = 1$ and $\gcd(c, 2, n) = 1$, we write an *odd vertex* $[a\sqrt{2}/c]_n$.
- If $[a/c\sqrt{2}]_n$ and $[b\sqrt{2}/d]_n$ with $a, b, c, d \in \mathbb{Z}/n\mathbb{Z}$ are two vertices of $\mathcal{M}_4(n)$, then those vertices are adjacent if and only if $ad - 2bc \equiv \pm 1 \pmod{n}$. Odd vertices are adjacent to even vertices, and vice-versa.
- The vertex valency of $\mathcal{M}_4(n)$ is n.
- Replacing 2 by 3 and 4 by 6 gives analogous results for $\mathcal{M}_6(n)$.

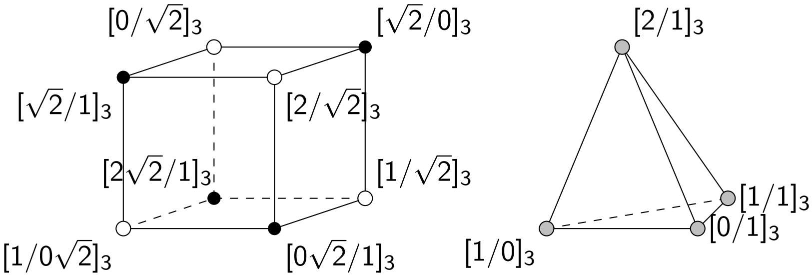


Figure: The Hecke map $\mathcal{M}_4(3)$ and the Farey map $\mathcal{M}_3(3)$.

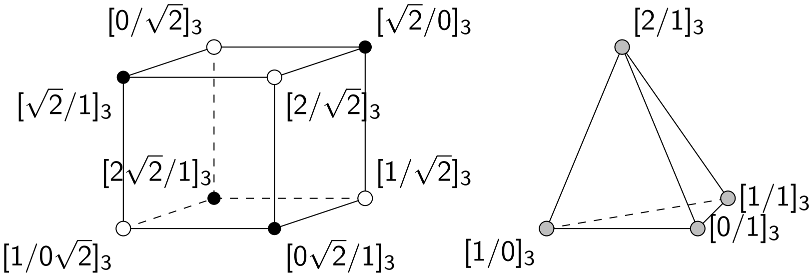
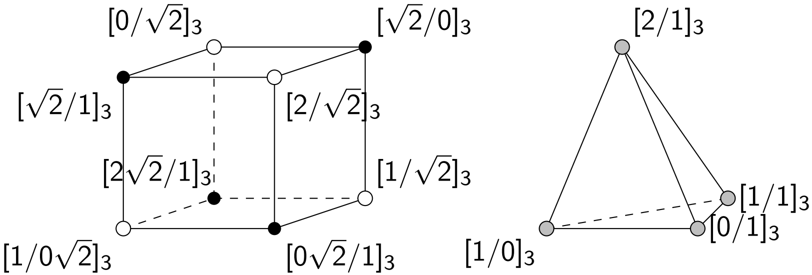


Figure: The Hecke map $\mathcal{M}_4(3)$ and the Farey map $\mathcal{M}_3(3)$.

Theorem

For odd n, $\mathcal{G}_4(n)$ is a double graph covering of $\mathcal{G}_3(n)$, and if n is not a multiple of 3, $\mathcal{G}_6(n)$ is a double graph covering of $\mathcal{G}_3(n)$.



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Theorem

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Theorem

The spectrum of $\mathcal{M}_4(n)$ for odd n, and that of $\mathcal{M}_6(n)$ for $3 \nmid n$, is the multiset $-\mathrm{sp}_3(n) \cup \mathrm{sp}_3(n)$, where $\mathrm{sp}_3(n)$ is the spectrum of $\mathcal{M}_3(n)$.

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THANK YOU