Rigid and flexible rod configurations

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Joint work with Signe Lundqvist and Lars-Daniel Öhman

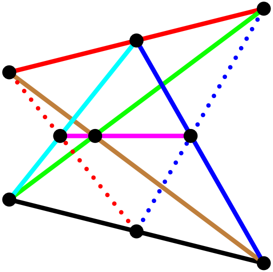
A configuration of points P and lines L in the plane is a set of lines together with a set of points in which the lines meet.

Traditionally there has been particular interest in the case when

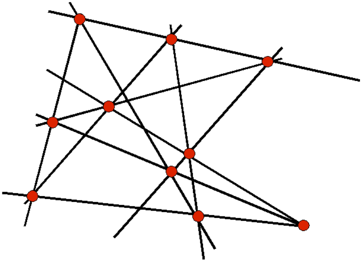
- all lines have k points, and
- all points are on r lines.

Denote $v = |P|$ and $b = |L|$, then $vr = bk$ is the number of incidences in the configuration.

If $k = r$, then the configuration is called balanced.

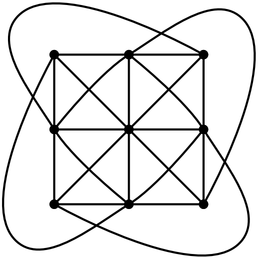


Pappus configuration

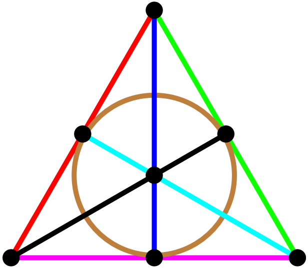


Desargues configuration

Many classical balanced configurations are related to projective theorems.



The affine plane over $\mathbb{F}_3$



The projective plane over $\mathbb{F}_2$

Many well-known configurations come from projective geometry over finite fields.

But wait a minute, some of those are not lines!

An incidence geometry of rank r consists of r sets $V_1, \dots, V_r$ and an incidence relation between them such that elements from the same V_i are not incident.

Certain abstract properties are usually imposed on the incidence relation to mimic geometric behaviour.

From a combinatorial point of view, configurations are incidence geometries of rank 2, they have elements of 2 types: **points** and **lines**.

The combinatorial abstract property of being a point and line configuration: every pair of points is in (i.e. joined by) at most one line.

If every point is on r lines and every line has k points, then we call such an incidence geometry a **combinatorial** (v, b, r, k)-**configuration**.

A geometric configuration is a geometric realisation of a combinatorial configuration.

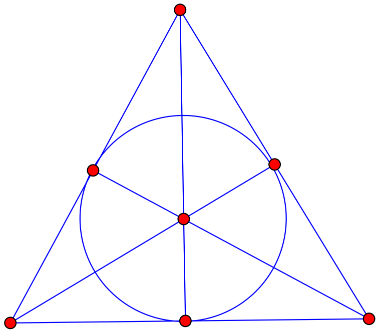
Just as a framework of a graph is a geometric realisation of the graph (as an incidence geometry if you want - graphs are also incidence geometries).

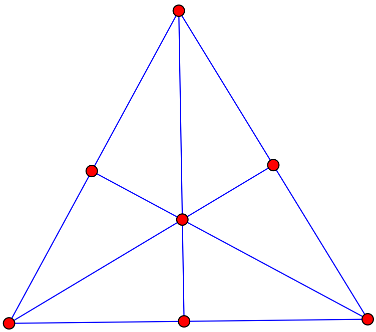
It is trivial to find a planar framework of a graph.

Just select any set of $|V|$ points and assign them to the vertices.

Theorem [conj. by Sylvester in 1893, Gallai 1944].
If X is a set of points in $\mathbb{R}^2$, not all on the same line, then there is a line containing exactly two points in X.

In the unique combinatorial 7_3-configuration (the Fano plane) every two points are joined by a line and every line is incident with three points. Also, the points are not all collinear. Therefore it cannot be realised in $\mathbb{R}^2$.

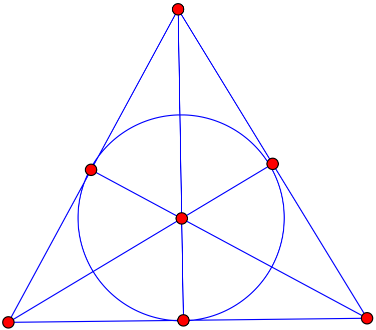


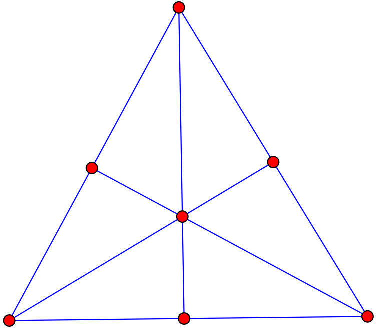


It is nice to know that things cannot get worse.

Theorem. [Steinitz, 1894]
Every connected combinatorial v_3-configuration can be realised as a point and line configuration in the Euclidean plane without additional incidences with at most one combinatorial incidence is not realised geometrically.

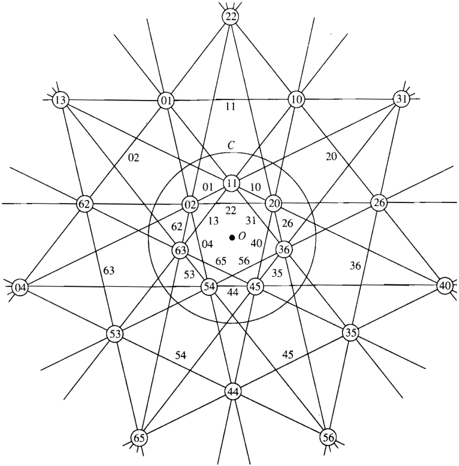
Moreover, the missing incidence can be chosen arbitrarily.





Geometric v_3-configurations have been studied since 1850.

The first drawing of a Euclidean v_4-configuration appeared not until 1990.



This configuration was first discovered by Klein in 1879.

The real realisation was provided by Grünbaum and Rigby in 1990.

Rigidity of graphs

A framework ρ of a graph $G = (V, E)$ in $\mathbb{R}^d$ is an assignment of a point $\rho(v) \in R^d$ to each vertex $v \in V$.

A continuous motion of a framework of G in $\mathbb{R}^d$ is a continuous motion of the points such that for all $(u, v) \in E$ the distance between any two vertices u and v is preserved.

Every framework is moved by the Euclidean motions: we call them the trivial motions.

A framework is rigid if the only continuous motions of the framework are the trivial motions.

Otherwise it is flexible.

An infinitesimal motion of a framework ρ in $\mathbb{R}^d$ is an assignment of a vector $m(v) \in \mathbb{R}^d$ to each vertex $v \in V$ so that

$$(\rho(v) - \rho(w))(m(v) - m(w)) = 0$$

for $(v, w) \in E$.

The linear part of a trivial motion gives a trivial infinitesimal motion.

A framework that has only the trivial infinitesimal motions is infinitesimally rigid.

Infinitesimal rigidity implies continuous rigidity.

A framework of a graph is generic if the set of coordinates is algebraically independent over $\mathbb{Q}$.

Lemma. Let $G = (V, E)$ be a graph. If there is an infinitesimally rigid framework of G in $\mathbb{R}^d$, then every generic framework of G in $\mathbb{R}^d$ is rigid.

For generic frameworks, continuous and infinitesimal rigidity are equivalent.

Therefore, we can talk about **the graph** as being rigid or flexible.

A graph G is minimally rigid/infinitesimally rigid if $G = (V, E)$ is rigid but the removal of any edge makes it flexible.

Theorem [Geiringer 1927, Laman 1970]
Let $G = (V, E)$ be a graph. Then G is minimally infinitesimally rigid in $\mathbb{R}^2$ if and only if

- $|E| = 2|V| - 3$, and
- $|E| \leq 2|V'| - 3$ for all subsets $E' \subseteq E$, where V' is the support of E'.

Graphs satisfying the second condition are called **independent**.

Graphs satisfying both conditions are called **Geiringer-Laman graphs**.

Combinatorial configurations

What about hypergraphs?

Hypergraphs are incidence geometries of rank two - combinatorial configurations.

We will use notation from incidence geometry: $S = (P, L, I)$ with "point" set P, "line" set L and incidence relation I.

We will allow lines to have different numbers of points and points to be on different number of lines.

A rod configuration is a configuration of points and lines in the Euclidean plane such that the lines moves as rods.

A motion of a rod configuration is a continuous motion of each point such that the distance between collinear points is preserved.

A rod configuration is rigid if the only motions it admits are the trivial motions.

Rod configurations can in general not be represented by a graph in generic position (if the incidence geometry is not a graph).

Therefore Geiringer's result cannot be applied directly.

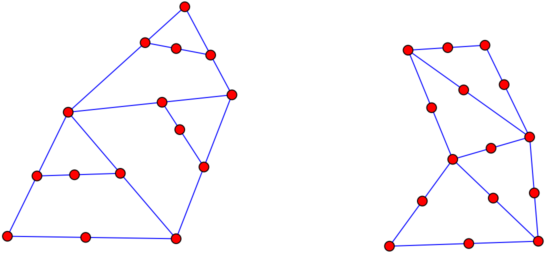
Definition.
Put a Geiringer-Laman graph instead of each rod. Say that the incidence geometry is **independent** if the resulting graph is independent.

Theorem [Whiteley 1989].
A rod configuration of an **independent** incidence geometry (P, L, I) is minimally infinitesimally rigid if and only if

- $2|I| = 3|L| + 2|P| - 3$, and
- $2|I'| \leq 3|L(I')| + 2|P(I')| - 3$

for all subset of incidences $I' \subseteq I$.

But . . . there are minimally infinitesimally rigid rod configurations that are not independent! Even more interesting: there are minimally rigid subconfigurations on the same number of points with different number of lines.



Two minimally rigid configurations on 15 points with 10 and 9 lines respectively.

Theorem [Jackson and Jordán, 2008].
A 2-regular hypergraph has an infinitesimally rigid rod configuration realization if and only if the graph obtained by replacing every hyperedge/line with a Geiringer-Laman graph is generically rigid.

In 3 dimensions the corresponding result can be applied to predict motions of molecules - it is called the “molecular conjecture”.

In 2011, Katoh and Tanigawa proved the molecular conjecture in arbitrary dimension.

Whiteley's result only applies to independent hypergraphs.

Jackson and Jordán's result only applies to 2-regular hypergraphs

(Note that the dual of a 2-regular hypergraph is a graph!)

So there was no result for determining the infinitesimal rigidity of rod configurations in general!

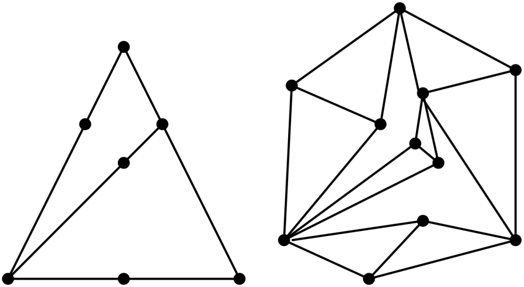
The **cone over a graph** (V, E) is the graph obtained by adding a cone vertex v and one edge (v, u) for all $u \in V$.

Coning [Whiteley 1983].
If G is a graph that is minimally infinitesimally rigid in $\mathbb{R}^d$, then the cone of G is minimally infinitesimally rigid in $\mathbb{R}^{d+1}$.

Example.
The minimally infinitesimally rigid graphs in $\mathbb{R}^1$ are the trees. The cone over a tree is therefore always minimally infinitesimally in $\mathbb{R}^2$.

Given a rod configuration we construct its cone graph by replacing every rod with the cone over a tree.

Every cone is a Geiringer-Laman graph, hence generically minimally infinitesimally rigid.



A linear realization ρ of a combinatorial configuration S in $\mathbb{R}^2$ is an assignment of a line in $\mathbb{R}^2$, with slope f_i and y-intercept h_i, to each element $l_i \in L$, and an assignment of point coordinates (x_j, y_j) to each element $p_j \in P$ such that if $(p_j, l_i) \in I$, then

$$f_i x_j + y_j + h_i = 0.$$

A linear realization is

- proper if distinct points are assigned distinct coordinates, and
- trivial if all points are assigned the same coordinates.

Fix the line slopes $\{f_i\}$ of a linear realization.

The solution of the $|I|$ equations of the form $f_i x_j + y_j + h_i = 0$ are the so-called **parallel redrawings** of ρ.

We call the $|I| \times (|L| + 2|P|)$-matrix encoding the I equations the **concurrence geometry matrix**.

The kernel of the concurrence geometry matrix has dimension at least 3 corresponding to the trivial parallel redrawings: dilation and translations. Other elements in the kernel are non-trivial parallel redrawings.

Parallel redrawings and infinitesimal motions are related by a duality defined by turning all vectors $\pi/2$.

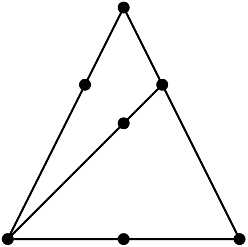
Let $S = (P, L, I)$ be an incidence geometry of rank two. We call a subset $I' \subseteq I$ **sharply independent** if

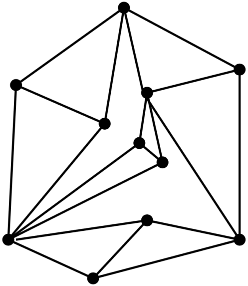
$$|J| \leq |M| + 2|Q| - 3$$

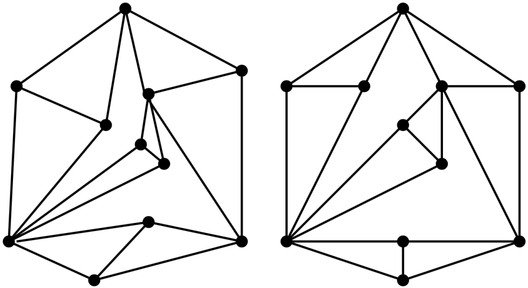
for any subset $J \subseteq I'$ where $Q \times M \subseteq P \times L$ is the support of J.

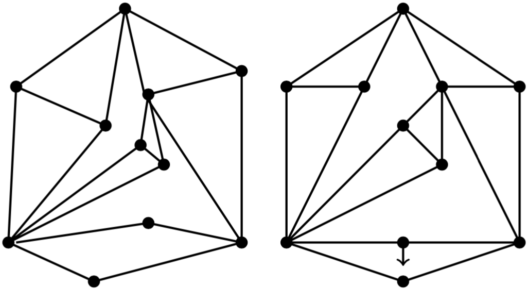
We say that a linear realization ρ of an incidence geometry $S = (P, L, I)$ is **regular** if the rows of the concurrence geometry matrix $M(S, \rho)$ corresponding to any sharply independent subset of I are independent.

Theorem [Lundqvist, S. and Öhman, 2022].
Let S be a combinatorial configuration realizable as a rod configuration in $\mathbb{R}^2$ such that the linear realization of its cone graph is **regular**. Then the infinitesimal rigidity of S is decided by the generic infinitesimal rigidity of any cone graph of S.



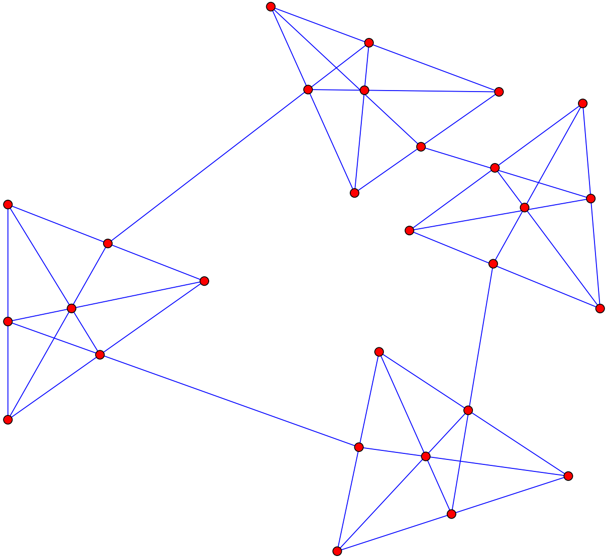


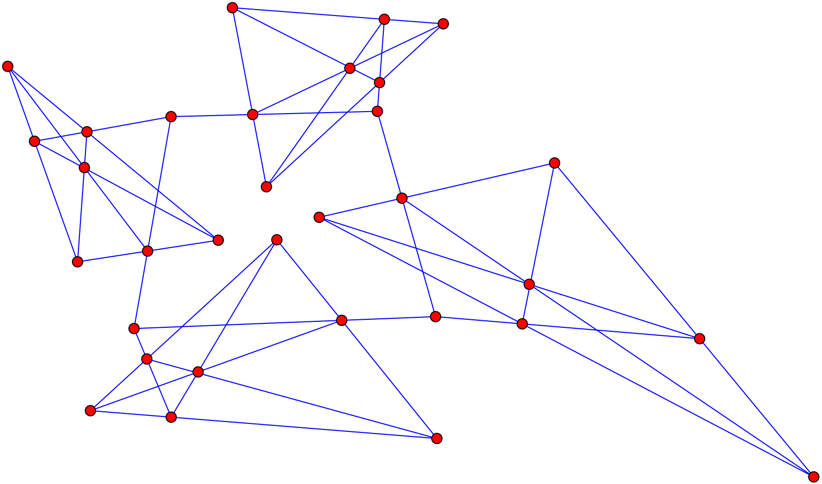




Flexible configurations

We can construct v_3-configurations that are flexible in regular position.



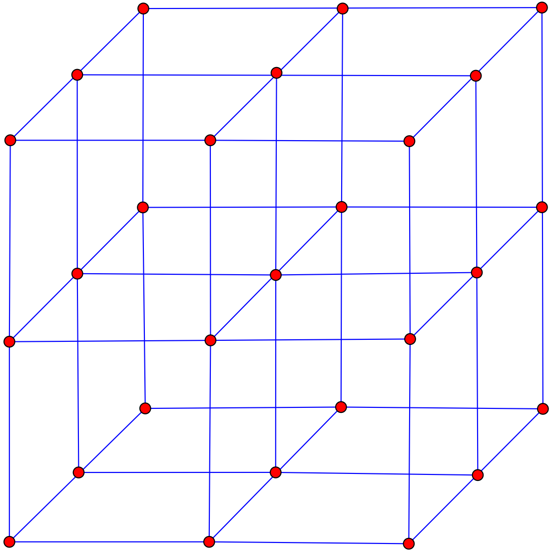


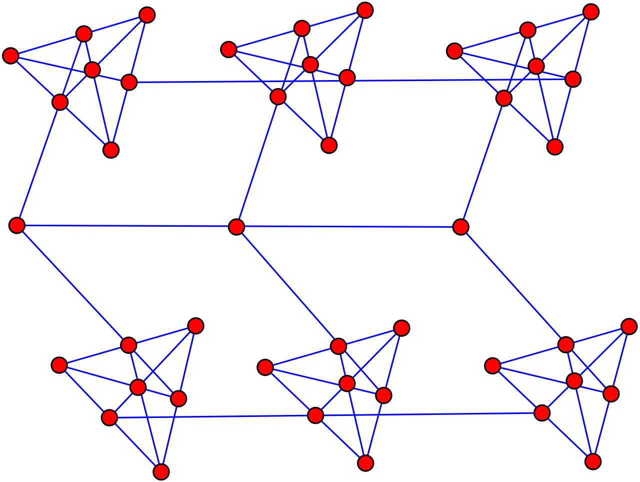
By combining the two previous examples we get regularly flexible v_3-configurations for $v = 7a + 8b$, whenever $a + b \geq 3$.

Because 7 and 8 are coprime, the largest integer not expressible as $7a + 8b$ with $a, b \in \mathbb{Z}$ is $7 \cdot 8 - 7 - 8 = 41$.

Therefore for all $v > 7 * 8 - 7 - 8 = 41$ there are regularly flexible v_3-configurations.

Just as is the case for graphs, rod configurations can be more flexible in special positions.





Some open problems:

- When is a rod configuration minimally infinitesimally rigid in the plane?
- When is a rod configuration infinitesimally rigid in higher dimensions?
- Inductive constructions of minimally infinitesimally rigid rod configurations.
- Constructions of more flexible configurations, exploring their properties.

References:

- S. Lundqvist, K. Stokes and L.-D. Öhman. Exploring the infinitesimal rigidity of planar configurations of points and rods. https://arxiv.org/abs/2110.07972.
- S. Lundqvist, K. Stokes and L.-D. Öhman. When is a planar rod configuration infinitesimally rigid?. https://arxiv.org/abs/2112.01960.

Thank you for listening!