Ramsey Numbers of Boolean Lattices

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Classical Ramsey Theory

Theorem
For all positive integers m and n there exists an integer N such that any blue-red coloring of K_N contains a monochromatic blue K_m or red K_n.

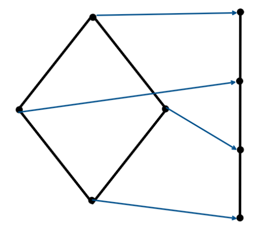
The smallest such N is called the Ramsey number $R(m, n)$.

In general we know $2^{n/2} \leq R(n, n) \leq 4^n$ and for fixed m, $R(m, n)$ is at most polynomial in n.

Basic Definitions

Definition
For partially ordered sets P and Q, we say that P is an *weak subposet* of Q if there is an injection $\phi : P \to Q$ such that $x <_P y \Rightarrow \phi(x) <_Q \phi(y)$ and an *induced subposet* if there is an injection such that $x <_P y \Leftrightarrow \phi(x) <_Q \phi(y)$.

For example the diamond poset on the left is a weak subposet of the 4-chain on the right, but not an induced subposet.



Poset Ramsey Number

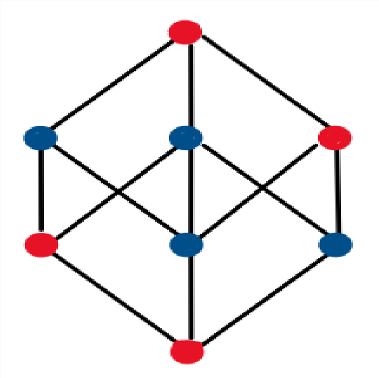
Definition
The Boolean lattice $Q_n = (2^{[n]}, \subseteq)$ is the poset of subsets of $[n] = \{1, 2, \ldots, n\}$ under the containment relation.

Definition
$R(Q_m, Q_n)$ is the minimum N such that any blue-red coloring of the elements of Q_N yields a blue Q_m or red Q_n as an induced subposet.

Definition
$R_w(Q_m, Q_n)$ is the minimum N such that any blue-red coloring of the elements of Q_N yields a blue Q_m or red Q_n as a weak subposet.

Example of a coloring

Clearly $R_w(Q_m, Q_n) \leq R(Q_m, Q_n)$ since every induced subposet is a weak subposet. Pictured below is an example of a coloring of Q_3 with a red induced Q_2.

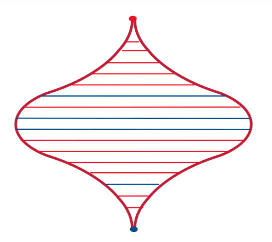


Diagonal Bounds

Theorem (Axenovich, Walzer 2017)
$2n \leq R(Q_n, Q_n) \leq n^2 + 2n.$

Theorem (Walzer (thesis))
$R(Q_n, Q_n) \leq n^2 + 1.$

A simple lower bound on $R(Q_m, Q_n)$ and $R_w(Q_m, Q_n)$ comes from a layered coloring in which all sets of a given cardinality are colored the same way.



Diagonal Bounds

Theorem (Lu, Thompson 2019)
$R(Q_n, Q_n) \leq n^2 - n + 2$.

Theorem (Cox, Stolee 2019)
$2n + 1 \leq R(Q_n, Q_n)$ *(probabilistic proof).*

Theorem (Bohman, Peng 2021)
$2n + 1 \leq R(Q_n, Q_n)$ *(direct construction).*

Theorem (Grósz, Methuku, T. 2021)
$2n + 1 \leq R_w(Q_n, Q_n)$ *(direct construction).*

Bounds for Q_2 versus Q_n

Theorem (Axenovich, Walzer 2017)
$n + 2 \leq R(Q_2, Q_n) \leq 2n + 2.$

Theorem (Lu, Thompson 2018)
$R(Q_2, Q_n) \leq \frac{5n}{3} + 2.$

Theorem (Grósz, Methuku, T. 2021)
$R(Q_2, Q_n) \leq n + O\left(\frac{n}{\log(n)}\right).$

Consequently, $R(Q_2, Q_n) = n(1 + o(1)).$

Theorem (Axenovich, Winter 2021)
$R(Q_2, Q_n) \geq n + \Omega\left(\frac{n}{\log(n)}\right).$

Consequently, $R(Q_2, Q_n) = n + \Theta\left(\frac{n}{\log(n)}\right).$

Theorem (Grósz, Methuku, T. 2021)
$R_w(Q_2, Q_n) = n + 3$ *(explicit, but rather complicated construction).*

Proof Sketch

Now I sketch the proof that $R(Q_2, Q_n) \leq n + O\left(\frac{n}{\log_2(n)}\right)$.

Let $k = cn/\log_2(n)$. And suppose by contradiction we have a coloring of Q_{n+k} avoiding a blue Q_2 and a red Q_n.

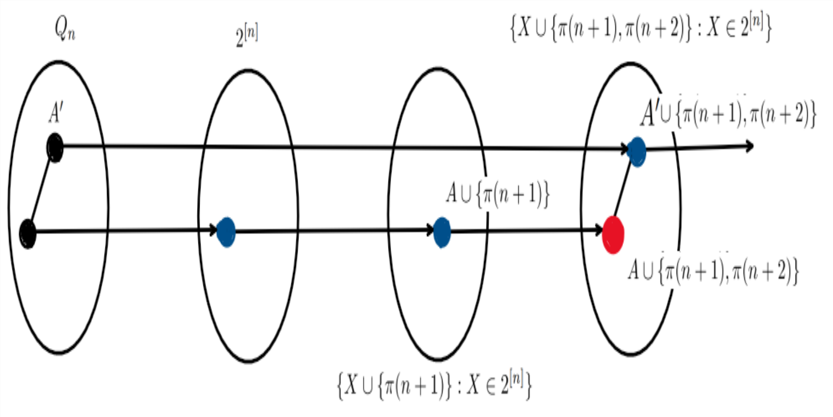
Using our assumption that there is no blue Q_2 in Q_{n+k}, we will try to find an embedding $\varphi : Q_n \to \text{Red Sets}\,(Q_{n+k})$.

Take an arbitrary permutation $\pi : [n+k]/[n] \to [n+k]/[n]$. We will search for an embedding of the form $\varphi : Q_n \to$
$\text{Red Sets}\left(\{X \cup Y : X \in 2^{[n]}, Y = \{\pi(n+1), \ldots, \pi(n+t)\}, t \leq k\}\right)$

The Embedding

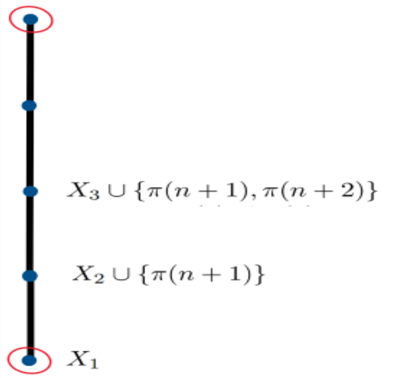
The embedding proceeds in a recursive way. We try to map the empty set to the empty set, unless the empty set is blue in the image. If the empty set is blue, then we append $\pi(n+1), \pi(n+2), \ldots$ and so on to $\varnothing$ until we get a red set.

Now suppose we are embedding A' and we have embedded each subset A of A'. Some of the A's may have been embedded by appending $\pi(n+1), \pi(n+2), \ldots$. Suppose for one of them we had to proceed until $\pi(n+t)$. Then we begin by trying to map A' to $A' \cup \{\pi(n+1), \ldots, \pi(n+t+1)\}$. If it is still blue, then we add $\pi(n+t+2)$ and so on.



If we succeed in embedding all of Q_n in the red sets in this way, then we have found a red Q_n. So assume we do not. Then we find a blue chain of length $k+1$ in a natural way.

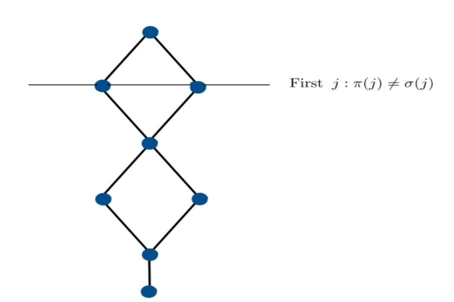
Map from permutations to pairs of sets



Record the top and bottom set of this blue $(k+1)$-chain. Then we obtain a map
$\{\pi : [n+k]/[n] \to [n+k]/[n] \text{ permutation}\} \to 2^{2(n+k)}$.
by $\pi \mapsto (A_{\text{top of chain}}, A_{\text{bottom of chain}})$.

Proof of Injectivity

This map is injective. Indeed, suppose σ and π are mapped to the same pair of sets. Let j be minimal such that $\sigma(j) \neq \pi(j)$. In the $(j+1)^{\text{st}}$ positions of the chains we have sets of the form $X \cup \{\sigma(n+1), \ldots, \sigma(n+j)\}$ and $Y \cup \{\pi(n+1), \ldots, \pi(n+j)\}$ (where $X, Y \subset [n]$) which are clearly incomparable. Thus along with the bottom and top sets of these chains we would have an induced diamond.



Contradiction as a consequence of injective map

Thus by injectivity of our map

$$\{\pi : [n+k]/[n] \to [n+k]/[n] \text{ permutation}\} \to 2^{2(n+k)},$$

we have $k! \leq 2^{2(n+k)}$ (recall $k = cn/\log_2(n)$). Using $k! \geq (k/e)^k = 2^{k(log_2 k - log_2 e)}$ and taking logarithms base 2, we have

$$k \log_2 k - k \log_2 e < 2(n+k) = 2n(1+o(1)). \tag{1}$$

However,

$$k \log_2 k > \frac{cn}{\log_2 n}(\log_2 c + \log_2 n - \log_2 \log_2 n - 1) = cn(1-o(1)). \tag{2}$$

Thus, by (1) and (2) we have a contradiction for any $c > 2$ when n is sufficiently large.

A Further Direction

It seems plausible that $R(Q_m, Q_n) = n(1 + o(1))$ for every fixed m as n tends to infinity.

A similar proof may yield such a result if we could answer the following question. Let $f(m, n)$ be the maximum number of maximal chains in $2^{[n]}$ with no induced Q_m. How fast does $f(m, n)$ grow?

In the following slide is a clip from the article showing the claim that we use to make the preceding proof sketch rigorous. The main idea is to define the embedding and record an ever growing blue chain in a recursive way and prove the properties inductively.

Claim 3. *Let* $\pi : [n+k]\setminus[n] \to [n+k]\setminus[n]$ *be a permutation. There exist* $\varphi : \mathcal{Q}_n \to \mathcal{R} \cup \{☹\}$ *(where ☹ is an arbitrary element, distinct from the members of* $\mathcal{R}$*, and used solely to indicate failure to produce an induced map into* $\mathcal{R}$*),* $\alpha : \mathcal{Q}_n \to \{0, 1, \ldots, k, k+1\}$ *and* $f : \mathcal{Q}_n \to \mathcal{C}^{\leq k+1}(\mathcal{B})$*, where* $\mathcal{C}^{\leq k+1}(\mathcal{B})$ *is the family of all chains of length at most* $k+1$ *in* $\mathcal{B}$*, with the following properties:*

P1. *If* $B, A \in \mathcal{Q}_n$ *and* $\varphi(B), \varphi(A) \in \mathcal{R}$*, then* $B \subsetneq A \iff \varphi(B) \subsetneq \varphi(A)$*. (This implies that if* $☹ \notin \operatorname{Im}\varphi$*, then* $\mathcal{Q}_n$ *is an induced subposet of* $\mathcal{R}$*.)*

P2. *If* $B \subseteq A \in \mathcal{Q}_n$*, then* $\alpha(B) \leq \alpha(A)$*.*

P3. *If* $\alpha(A) = k+1$*, then* $\varphi(A) = ☹$*. Otherwise* $\varphi(A) \cap [n] = A$*, and* $\varphi(A) = A \cup \{\pi(n+1), \pi(n+2), \ldots, \pi(n+\alpha(A))\}$*.*

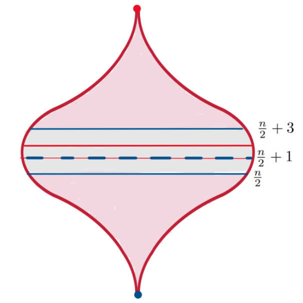
P4. *For every* $A \in \mathcal{Q}_n$*,* $f(A) = \big(f(A)_0, f(A)_1, \ldots, f(A)_{\alpha(A)-1}\big)$ *is a chain in* $\mathcal{B}$ *of length* $\alpha(A)$ *with the property that* $f(A)_i \setminus [n] = \{\pi(n+1), \pi(n+2), \ldots, \pi(n+i)\}$*.*

P5. *If* $A \in \mathcal{Q}_n$ *such that* $1 \leq \alpha(A) \leq k$*, then* $f(A)_{\alpha(A)-1} \subseteq \varphi(A)$*. (In fact this implies that* $f(A)_{\alpha(A)-1} \subsetneq \varphi(A)$*, since the elements of* $f(A)$ *are in* $\mathcal{B}$*, while* $\varphi(A)$ *is in* $\mathcal{R}$*. We do not use this observation.)*

Lower bounds

Now we briefly discuss the lower bounds. Here we sketch an argument for $R(Q_2, Q_n) \geq n+3$. With much more care it can be shown that $R_w(Q_2, Q_n) \geq n+3$.

Assume $N = n+2$ is even. The idea is to take all of the levels $n/2$ and $n/2+3$ in blue as well as some additional blue sets not forming a diamond on level $n/2+1$ in order to block all red Q_n's.



Lower Bounds

We select blue sets of size $n/2+1$ in a greedy way to satisfy some properties:

- They have pairwise symmetric difference at least 4.
- For each $x, y \in [n+2]$ pair, we select a set that contains x and not y.

Based on the total number of sets in $\binom{[n+2]}{n/2+1}$ we can select sets in a greedy way to satisfy these properties. The first property ensures that we have no blue diamond. Indeed, any such diamond must include two sets on level n/2+1 and a set on level n/2 in their intersection, but this is impossible if the two sets have symmetric difference 4 or more.

Lower bounds

The second property is designed to block any proposed embedding of a red Q_n. Checking the details here is complicated, but the idea is that in such an embedding ϕ, since two potential levels (n/2 and n/2+3) are omitted we deduce that singletons must map to singletons. So each potential ϕ determines two elements $x, y \in [n+2]$ which are not singletons in the image of ϕ.

An analysis shows that then no set of size $< n/2$ has an image under ϕ which uses x or y. The sets of size $n/2+1$ or $n/2+2$ use exactly one of x or y (same for all), and the sets of size $> n/2+4$ use both. Suppose in our embedding we use x in the middle sets. Then for $A' \cup \{x\}$ in our blue family, there can be no preimage under ϕ.

Thank you for your attention!