Turán problems in digraphs

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What is a graph?

A graph G consists of a collection $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of vertices called *edges*.

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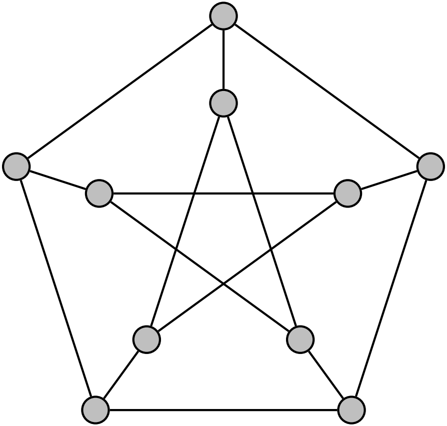
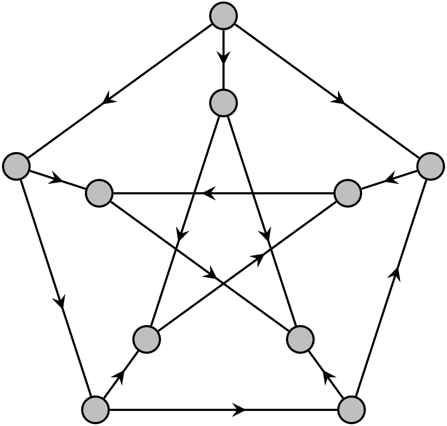


Figure 1: The Petersen graph

What is a digraph?

A directed graph G consists of a collection $V(G)$ of *vertices* and a set $A(G)$ of ordered pairs of vertices called *arcs*.



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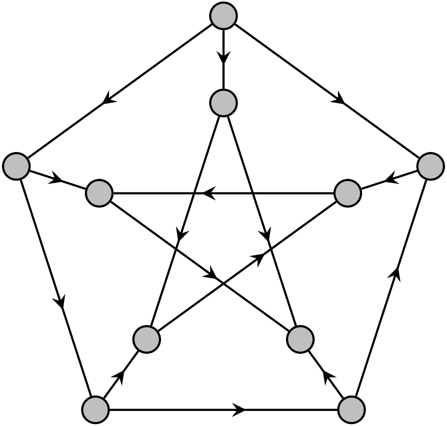
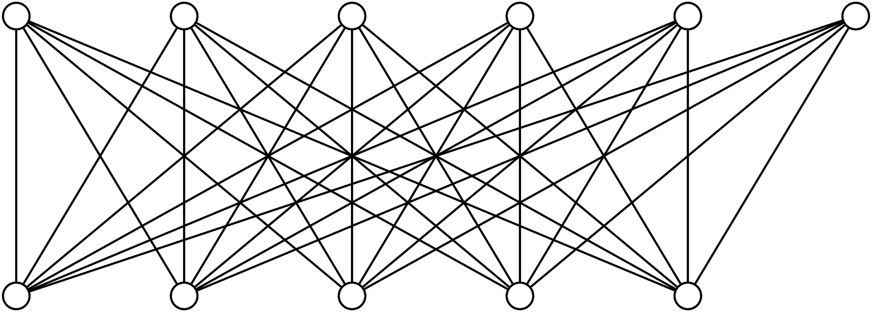


The number of arcs coming from a vertex x is the *out-degree* $d^+(x)$.
The set of *out-neighbours* is $N^+(x)$.

How many edges can we have without triangles?



Therefore we can have at least $\lfloor \frac{n^2}{4} \rfloor$ edges.

How many edges can we have without triangles?

Suppose that there is a graph with order n, size $m > \lfloor \frac{n^2}{4} \rfloor$ and no triangles. Let G be such a graph with smallest order and let uv be an edge of G. u and v have no common neighbours, so $d(u) + d(v) \leq n$.

Consider the graph $G' = G - \{u, v\}$. G' is triangle-free and has size

$$m' > m - (d(u) + d(v)) + 1 > \frac{n^2}{4} - n + 1 = \frac{(n-2)^2}{4},$$

a contradiction.

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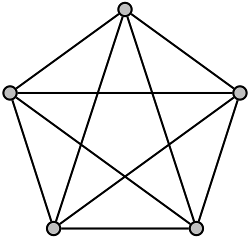
Theorem, Mantel, 1907

The largest size of a triangle-free graph with order n is $\lfloor \frac{n^2}{4} \rfloor$ and for order n the extremal graphs are complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

A triangle is...

A clique is a graph with order n and all $\binom{n}{2}$ possible edges.



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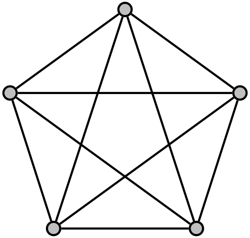
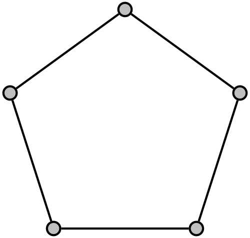


A cycle



What if we forbid cliques?

Turán's Theorem

The number of edges of a K_{r+1}-free graph H is at most $(1 - \frac{1}{r})\frac{n^2}{2}$.

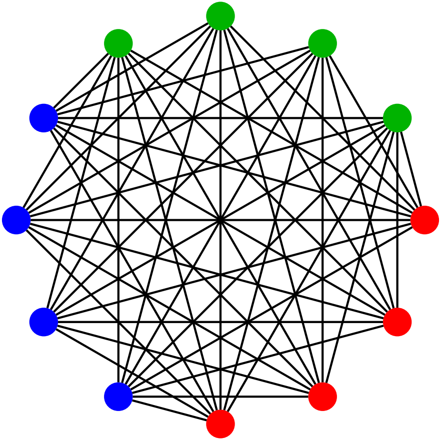


Figure 2: A Turán graph

A Turán problem for cycles

Question, Erdős

What is the maximum size of a graph with order n and no cycles of length $\leq r$?

Erdős conjectured that for $r = 4$ the answer is $(\frac{1}{2} + o(1))^{\frac{3}{2}} n^{\frac{3}{2}}$. If we denote the extremal size by $f(n)$, then it is only known that

$$\frac{1}{2\sqrt{2}} \leq \liminf_{n\to\infty} \frac{f(n)}{n^{\frac{3}{2}}} \leq \limsup_{n\to\infty} \frac{f(n)}{n^{\frac{3}{2}}} \leq \frac{1}{2}.$$

Finding exact values for given n and r is a difficult open problem.

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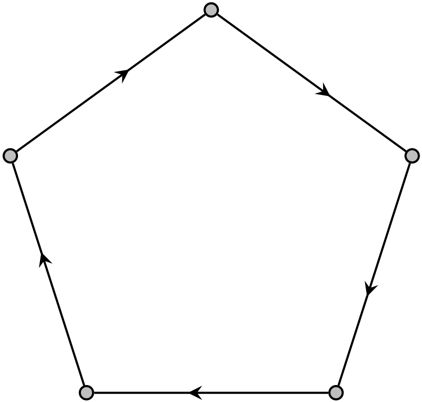
What can we say about directed graphs?

Excluding directed cycles



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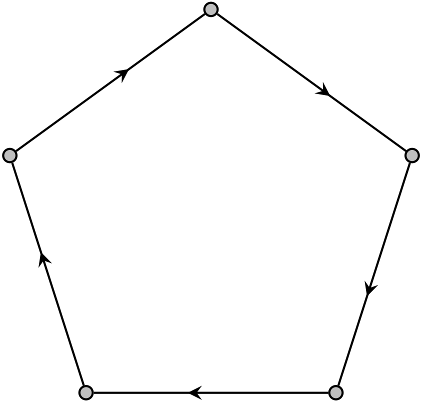


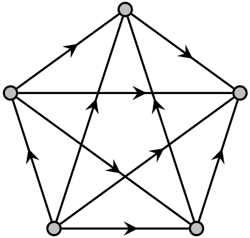
Question

What is the largest size of a strongly connected digraph with girth $\geq g$?

Excluding cycles in digraphs

We must include the restriction of strong connectivity as the acyclic tournament with order n has $\binom{n}{2}$ arcs but no cycles.

I can orient the edges in such a way that I get a digraph of order n, with $n(n-1)/2$ arcs, and **no directed cycles at all**.



Solution for cycles

This question was completely solved by Bermond et al. (see ‘Girth in digraphs’).

Theorem

Let D be a strong digraph of order n, size m and girth g. Let $k \geq 2$. Then

$$m \geq \frac{1}{2}(n^2 + (3-2k)n + k^2 - k)$$

implies that $g \leq k$. This expression is best possible.

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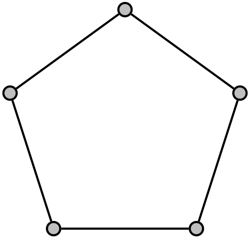
This means that, asymptotically speaking, a strong digraph can have large girth and ‘almost all’ possible arcs present!

What is a k-geodetic digraph?

A graph has girth $\geq g$ if and only if it is $\frac{g-1}{2}$-geodetic.



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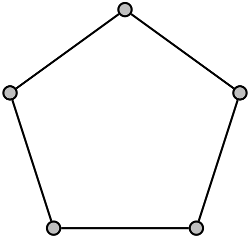
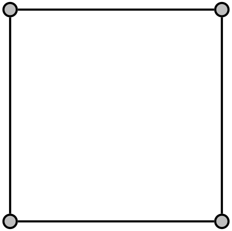




What is a k-geodetic digraph?

Definition

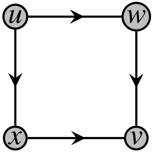
For $k \in \mathbb{Z}$, a digraph is

k-*geodetic* if there do not exist vertices u, v with two distinct directed paths of length $\leq k$ between them and

$(k + \frac{1}{2})$-geodetic if for any pair of vertices u, v and distinct paths P, Q from u to v, both with length $\leq k + 1$, then both P and Q have length exactly $k + 1$.

Examples

This digraph is not 2-geodetic.



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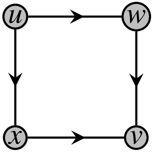
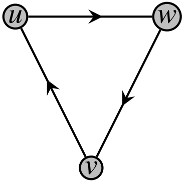


This digraph is not 3-geodetic.



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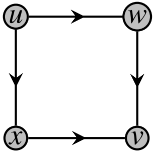

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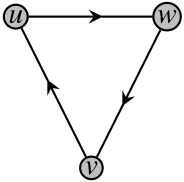
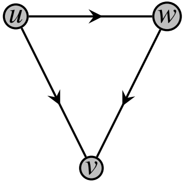


This digraph is not $1\frac{1}{2}$-geodetic.



Why are such digraphs interesting?

The **degree/geodecity problem** asks for the smallest possible order of a k-geodetic digraph with minimum out-degree d. It is known that the order n of such a digraph is bounded below by the directed Moore bound

$$n \geq M(d,k) = 1 + d + d^2 + \cdots + d^k.$$

The degree/geodecity problem is a generalisation of the undirected degree/girth problem.

The geodetic girth of a digraph G is the largest k such that G is k-geodetic. As an undirected graph has girth $\geq 2k+1$ if and only if it is k-geodetic (with suitable changes made to the definition) the geodetic girth of a digraph can be viewed as a 'girth-like' parameter.

An example of a cage

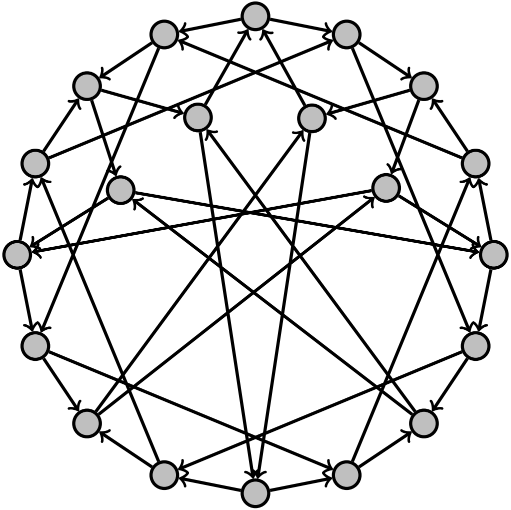


Figure 3: A smallest possible 3-geodetic digraph with out-degree 2

Regular digraphs

Theorem, Ustimenko et al.

For fixed k, the largest possible size of a diregular k-geodetic digraph with order n is asymptotic to $n^{1+\frac{1}{k}}$ as $n \to \infty$. This bound is met asymptotically by the permutation digraphs $P(d,k)$.

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The order n of a k-geodetic digraph with minimum out-degree d is bounded below by the directed *Moore bound* $M(d,k) = 1 + d + d^2 + \cdots + d^k$. Hence $n \geq d^k$ and, rearranging, $d \leq n^{1/k}$. The size m of an out-regular k-geodetic digraph G with order n thus satisfies $m = nd \leq n^{\frac{k+1}{k}}$.

A permutation digraph

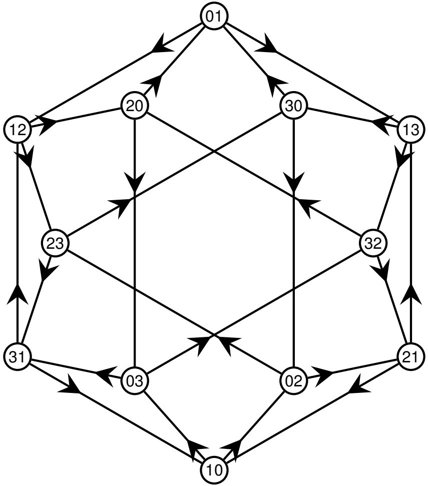

Figure 4: $P(2,2)$

Problem statement

Definition

Let $ex(n;k)$ be the largest possible size of a k-geodetic digraph with order n.

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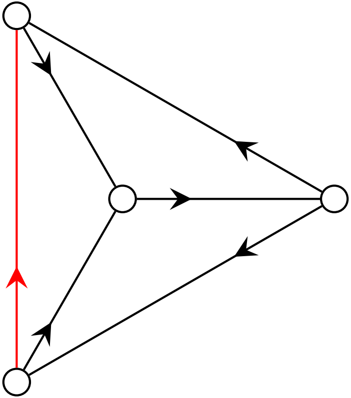

Theorem

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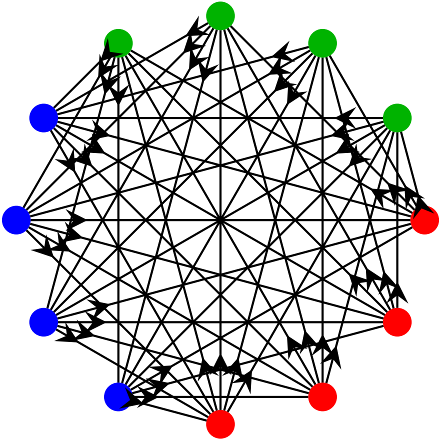
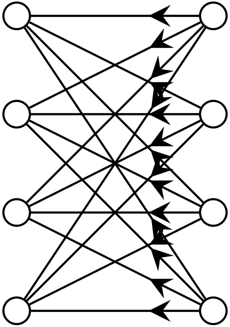
$ex(n; \frac{3}{2}) = t_{n,3}$, where $t_{n,3}$ is the size of the 3-partite Turán graph with order n.



What about $k \geq 2$?

We can easily obtain a lower bound of $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$ by taking the complete bipartite graph $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and orienting all arcs towards the same partite set.



Counting Lemma

Lemma

For any $m \leq n-1$ we have $ex(n;k) \leq \frac{n(n-1)}{m(m-1)} ex(m;k)$.

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Proof

We count the pairs (F,e), where F is a subset of m vertices and e is an arc with both end-points in F. Let F be any subset of m vertices. In the induced subdigraph there can be at most $ex(m;k)$ arcs. Therefore there are at most $\binom{n}{m} ex(m;k)$ such pairs. For each arc e there are $\binom{n-2}{m-2}$ subsets containing the endpoints of e, so it follows that

$$ex(n;2)\binom{n-2}{m-2} \leq \binom{n}{m} ex(m;k).$$

Rearranging yields the result.

Proof 1

Theorem

For all $n \geq 4$, $n \geq k \geq 2$ we have $ex(n;k) = \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$.

Let $k = 2$. Let $n \geq 5$ and assume that the theorem is true for $n - 1$.

If $n = 2r$, by the counting lemma and induction

$$ex(2r;2) \leq \frac{2r(2r-1)}{(2r-1)(2r-2)} ex(2r-1;2) = \frac{2r(2r-1)}{(2r-1)(2r-2)} r(r-1) = r^2$$

as required.

If $n = 2r + 1$, the counting lemma with $m = 2r$ gives

$$ex(2r+1;2) \leq \frac{2r(2r+1)}{2r(2r-1)} ex(2r;2) = \frac{2r(2r+1)}{2r(2r-1)} r^2 < r^2 + r + 1,$$

so again the necessary inequality follows.

Proof 2

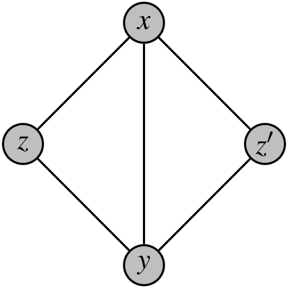

Figure 6: There is no 2-geodetic orientation of K_4^-

Proof 2

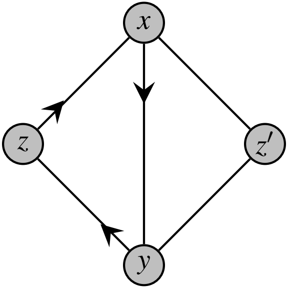

Figure 7: There is no 2-geodetic orientation of K_4^-

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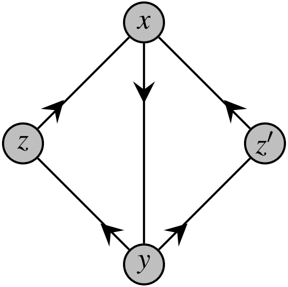
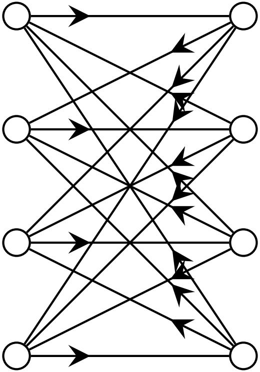


Figure 8: There is no 2-geodetic orientation of K_4^-

Theorem

For $n \geq 4$ and $k \geq 2$ we have $ex(n;k) = \lfloor \frac{n^2}{4} \rfloor$ and for $n \geq 7$ the extremal 2-geodetic digraphs are orientations of $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ with all arcs oriented in the same direction, except for a matching that is oriented in the opposite direction.

What if we require strong connectivity?



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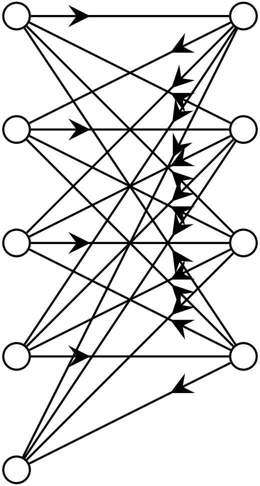

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For $k \geq 2$ and $n \geq k$ let $ex^*(n;k)$ be the largest size of a strongly connected k-geodetic digraph with order n.

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For $k \geq 2$ and $n \geq k$ let $ex^*(n;k)$ be the largest size of a strongly connected k-geodetic digraph with order n.

We know that $ex^*(2r;2) = r^2$, but $ex(2r+1;2) < r^2 + r$.

A construction with $n = 2r + 1$ and $r^2 + 2$ arcs

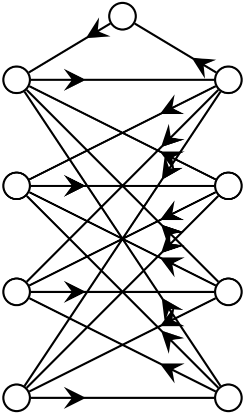


Figure 9: A strongly connected digraph with $n = 2r + 1$ and $m = r^2 + 2$ (for $r = 4$)

In fact this is best possible!

Proof sketch

Theorem

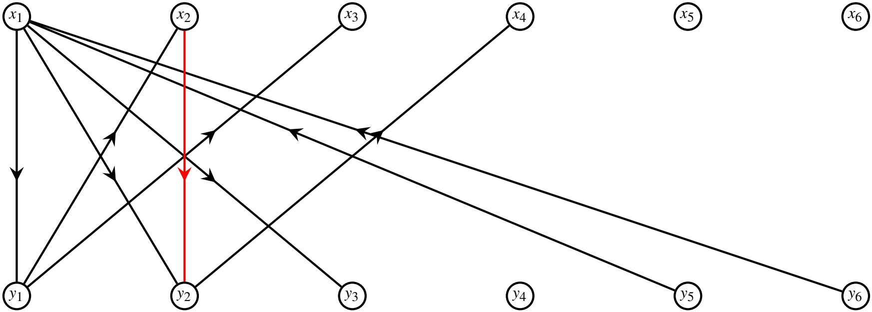
$ex^*(2r;2) = r^2$ and $ex^*(2r+1;2) = r^2+2$.

Let G be k-geodetic digraph with order $n = 2r+1$ and size $m \geq r^2+3$. Let H be its underlying undirected graph.

A stability result tells us that H is either bipartite or contains a triangle.

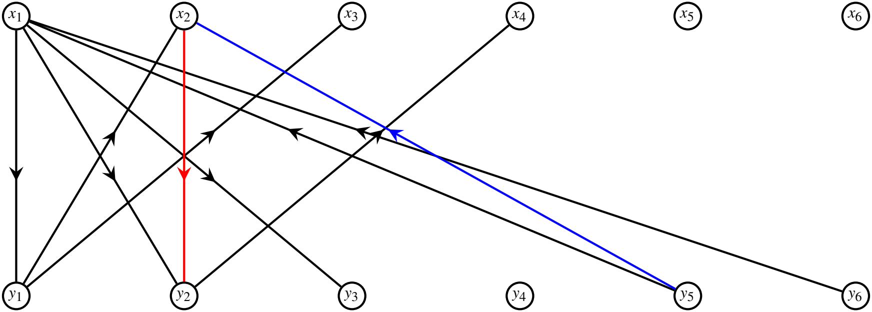
Suppose that H contains a triangle T. As G is diamond-free, deleting T removes at most $2r+1$ arcs. Then the size of $G-T$ is $> (r-1)^2$. Thus H is bipartite.

Proof sketch



If there is an arc from x_2 to $N^+(x)$, there are no arcs from $N^-(x)$ to x_2 and there can be at most one arc from x_2 to $N^-(x)$, so there are at least $d^-(x) - 1$ arcs missing between x_2 and $N^-(x)$.

Proof sketch



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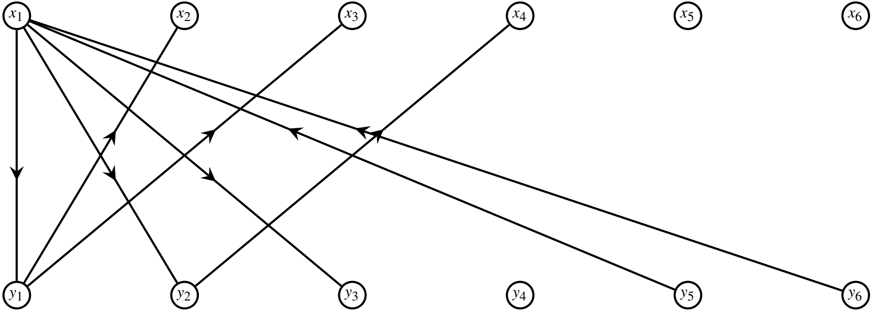

If there are no arcs from x_2 to $N^+(x)$, then there are at least $d^+(x) - 1$ arcs missing between x_2 and $N^+(x)$. Repeating this for all members of $N^2(x)$ shows that there are at least $\max\{|N^{+2}(x)|, |N^{-2}(x)|\}(\min\{d^+(x), d^-(x)\} - 1)$ missing edges.

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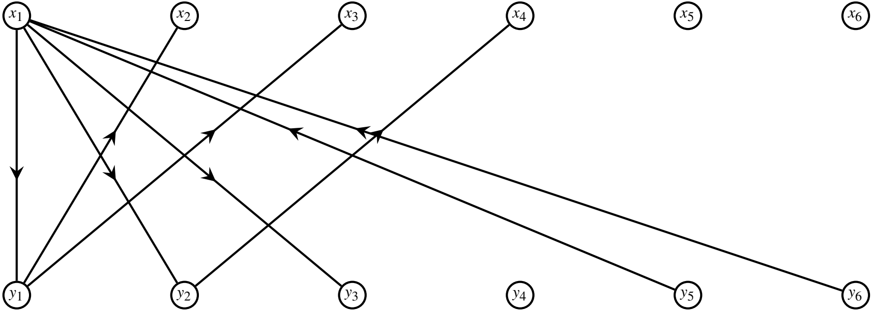

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Classification for $k = 2$

This analysis allows us to classify all strong 2-geodetic digraphs with order $n = 2r + 1$ and size $m = r^2 + 2$. Examples of these digraphs are shown on the following slides.

Theorem

If G is a 2-geodetic digraph with order $n = 2r + 1$, size $m = r^2 + 2$ and no sources or sinks, then G is either isomorphic to one of $A_r, B_{r,0}, B_{r,r-1}, C_r$ or D_r or is isomorphic to a member of the family $B_{r,t}, B'_{r,t}$ for some $1 \leq t \leq r - 2$. The digraphs in this list are mutually non-isomorphic and so there are $2r + 1$ distinct solutions up to isomorphism.

Strong digraphs with $n = 2r + 1$, $m = r^2 + 2$

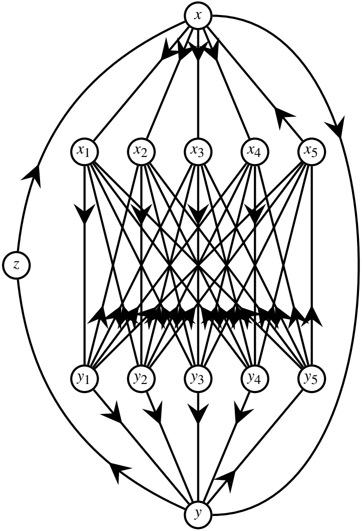


Figure 10: A_6

Strong digraphs with $n = 2r + 1$, $m = r^2 + 2$

This digraph is a member of a family of $t - 1$ solutions.

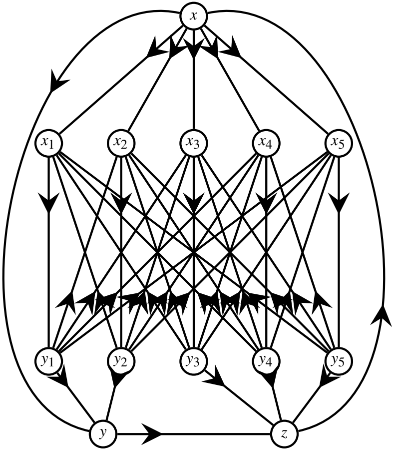


Figure 11: $B_{6,2}$

Strong digraphs with $n = 2r + 1$, $m = r^2 + 2$

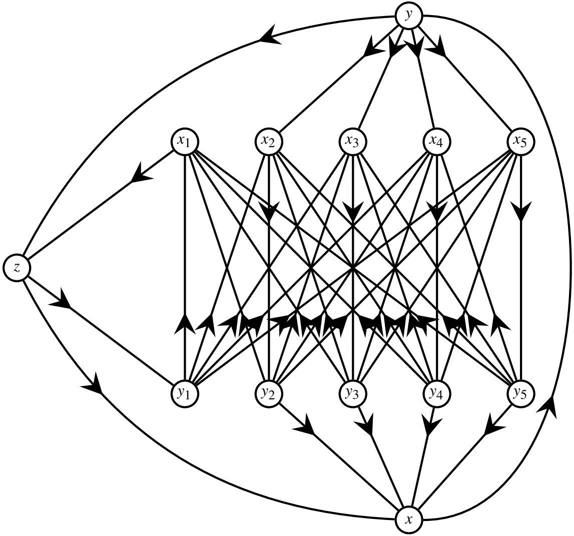


Figure 12: C_6

Strong digraphs with $n = 2r + 1$, $m = r^2 + 2$

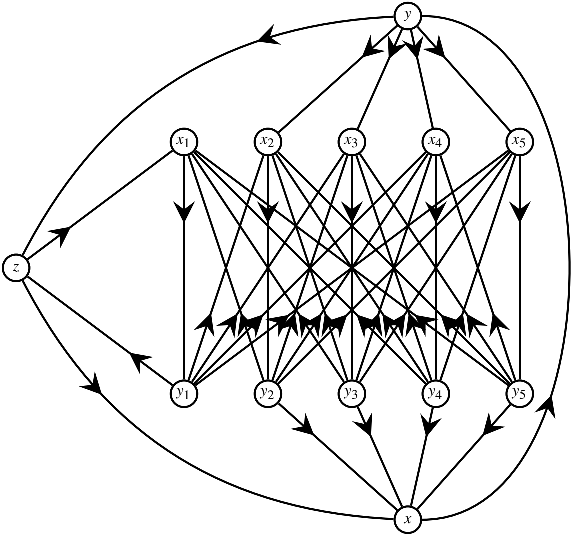


Figure 13: D_6

What about larger k?

Lemma

$\frac{n^2}{k^2} \le ex^*(n;k) \le \frac{n^2}{k}$

Suppose that there is a vertex v with out-degree $\ge \frac{n}{k}$. Then each set $N^t(v)$ has $\ge \frac{n}{k}$ members for $1 \le t \le k$, giving $\ge n+1$ distinct vertices.

Take the orientation of the complete bipartite graph $K_{r,r}$ with a perfect matching oriented in one direction and all other arcs in the opposite direction. Expand the arc in the perfect matching into paths of length $k-1$. This yields a strongly connected k-geodetic digraph $G_{k,r}$ with order kr, $r \ge 2$.This gives a lower bound of order $\frac{n^2}{k^2}$.

Example

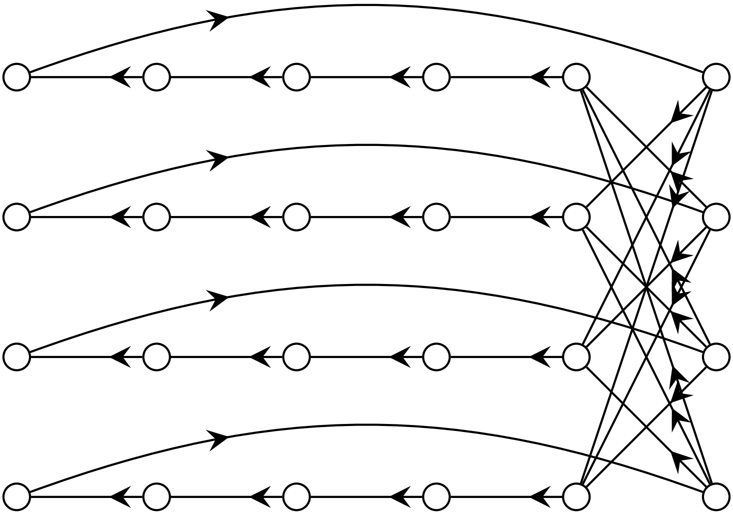


Figure 14: $G_{k,r}$ for $k = 6, r = 4$

More generally...

Let the quotient and remainder when n is divided by k be r and s respectively, i.e. $n = kr + s$. We assume that $s \leq r$.

Form the digraph $G(n, k)$ as follows. The vertex set of $G(n, k)$ consists of vertices $u_{i,j}$ for $1 \leq i \leq r$ and $1 \leq j \leq k$, as well as s further vertices $v_1, v_2, \ldots v_s$.

We define the adjacencies of $G(n, k)$ as follows.

i) $u_{i,j} \to u_{i,j+1}$ for $1 \leq i \leq r$ and $1 \leq j \leq k - 1$

ii) $u_{i,k} \to v_i$ for $1 \leq i \leq s$

iii) $u_{i,k} \to u_{j,2}$ for $s + 1 \leq i \leq r$ and $1 \leq j \leq s$

iv) $u_{i,k} \to u_{i',1}$ for $s + 1 \leq i, i' \leq r$ and $i \neq i'$

More generally...

This digraph is k-geodetic and has size

$$m = rs + (k-1)r + s + (r-s)(r-1) = r^2 + (k-2)r + 2s.$$

If $r+1 \leq s \leq k-1$, then we have $\lfloor \frac{n}{k} \rfloor \leq k-2$, which is equivalent to $n \leq k^2 - k - 1$. Therefore these digraphs will certainly exist for $n \geq k^2 - k$. The arcs in part iii) can also be directed to $u_{j,2}$; combined with taking the converse of the resulting digraphs, this generates several isomorphism classes.

Let's see an example.

Example

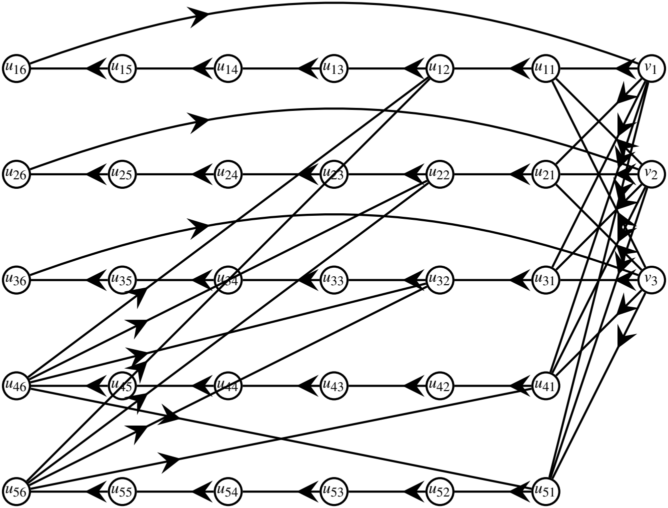


Figure 15: $G(33,6)$

Computational results

n/k	3	4	5	6
7	8			
8	10			
9	12	10		
10	14	12		
11	16	14	12	
12	20	15	14	
13	22	17	15	14
14	24	19	17	16
15		21	18	17
16			20	19
17			22	20
18				21
19				23

Table 1: $ex^*(n;k)$ for some small values of n and k

Conjecture

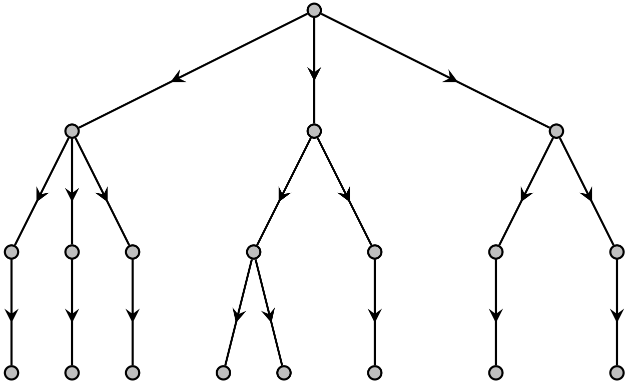
Conjecture

If $n \geq k+1$ and $n \leq (k+1)\lfloor \frac{n}{k} \rfloor$ (in particular for $n \geq k^2 - k$),

$$ex^*(n;k) = \lfloor \frac{n}{k} \rfloor^2 - (k+2)\lfloor \frac{n}{k} \rfloor + 2n.$$

Also for $r \geq 2$ the underlying graph of $G(kr, k)$ is the unique graph with largest size to have a strongly connected k-geodetic orientation.

$k = 3$ argument



If the max. degree is x, the number of arcs is bounded above asymptotically by $(n-2x)x = -2x^2 + nx$. This has its maximum at $x = \frac{n}{4}$, giving $ex^*(n;3) \leq \frac{n^2}{8}$. A more delicate argument (due to Tompkins and Hendrey) confirms that $ex^*(n;3) \sim \frac{n^2}{9}$.

Larger k: Main Theorem

Theorem, Győri and Salia

$ex^*(n,k) \leq \frac{2n^2}{k^2+k} + O(n)$.

Let G be a k-geodetic digraph with order n and size m and let $m_t(x)$ be the number of directed paths of length $t \geq 0$ starting at the vertex x.

It follows by induction that there are at least $(m-n+1)t$ directed paths of length t. Induction step:

$$\sum_{v \in V(D)} d^-(v) m_{t-1}(v) \geq \sum_{v \in V(D)} (d^-(v) + m_{t-1}(v) - 1) \geq m - n + \sum_{v \in V(D)} m_{t-1}(v)$$

which at least $(m-n+1)t$. By k-geodecity for any ordered pair u, v there is at most one u, v-path with length $\leq k$. Summing over all paths,

$$\sum_{t=1}^{k} (m-n+1)t = \frac{k(k+1)}{2} m + O(n) < n^2.$$

$k = \frac{5}{2}$

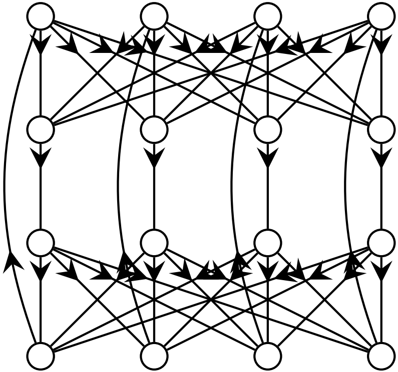


Figure 16

Theorem

$ex^*(n; \frac{5}{2}) \geq n^2/8 + O(n)$.

Generalised Turán problems

Definition

For any digraph Z and $k \geq 2$ we denote the largest number of copies of Z in a k-geodetic digraph by $ex(n; Z; k)$.

Generalised Turán problems

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For any digraph Z and $k \geq 2$ we denote the largest number of copies of Z in a k-geodetic digraph by $ex(n; Z; k)$.

Theorem

$ex(n; C_3; 2) = \frac{1}{3}n^{3/2} + O(n^{\frac{1}{2}})$.

Theorem

For even ℓ, $ex(n; P_\ell; 2) = n^{(\ell/2)+1} + O(n^{\ell/2})$. For odd paths $ex(n; P_3; 2) = (n/2)^3 + O(n^2)$.

Conjecture

$ex(n; C_{k+1}; k) \sim \frac{1}{k+1}n^{\frac{k+1}{k}}$ and $ex(n; P_l; 2) \sim (n/2)^{(l+3)/2}$.

Some open problems

What is $ex^*(n; k + \frac{1}{2})$ for $k \geq 2$?

Can we get rid of the factor of two?

What are the extremal digraphs?

What about directed hypergraphs?

What is the largest number of cycles/paths/hooves?

Some open problems

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Thank you!