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On some optimisation problems for homogeneous algebraic graphs.

Vasyl Ustimenko [1,2]

[1]Royal Holloway University of London

[2]Institute of telecommunication and global information space, NAS of Ukraine

E - mail: Vasyl.Ustymenko@rhul.ac.uk,

Abstract. Homogeneous algebraic graphs defined over arbitrary field are classical objects of Algebraic Geometry. This class includes geometries of Chevalley groups $A_2(F)$, $B_2(F)$ and $G_2(F)$ defined over arbitrary field F. Assume that codimension of homogeneous graph is the ratio of dimension of variety of its vertices and the dimension of neighbourhood of some vertex. We evaluate minimal codimension $v(g)$ and $u(h)$ of algebraic graph of prescribed girth g and cycle indicator. Recall that girth is the size of minimal cycle in the graph and girth indicator stands for the maximal value of the shortest path through some vertex. We prove that for even h the inequality $u(h) \leq (h - 2)/2$ holds. We define a class of homogeneous algebraic graphs with even cycle indicator h and codimension $(h - 2)/2$. It contains geometries $A_2(F)$, $B_2(F)$ and $G_2(F)$ and infinitely many other homogeneous algebraic graphs.

1.1. On special optimisation problems for homogeneous algebraic graphs.

Let us start from the concept of homogeneous algebraic graph.

Let ***F*** be a field .

Recall that a projective space over ***F*** is a set of elements constructed from a vector space over ***F*** such that a distinct element of the projective space consists of all non-zero vectors which are equal up to a multiplication by a non-zero scalar.

Its subset ***Q*** is called a quasiprojective variety, if it is the set of all solutions of some system of homogeneous polynomial equations and inequalities.

If F is a field than we can define a dimension $\dim Q$ of Q (3 equivalent definitions are known, for instance it can be defined via minimal Grobner basis)

An algebraic graph ψ over F consists of two things: the vertex set Q being a quasiprojective

variety over F of non-zero dimension and the edge set being a quasiprojective variety ψ in $Q \times Q$

such that (x, x) is not element of ψ for each x from Q,

and $x \ \psi \ y$ implies $y \ \psi \ x$ ($x \ \psi \ y$ means (x, y) is an element of ψ.

The graph ψ is homogeneous (or N-homogeneous), if for each vertex w from Q, the set $\{x \mid w \ \psi \ x\}$ is isomorphic a quasiprojective variety ***M(w)*** over ***F*** of a non-zero dimension constant parameter ***N*****.**

We further assume that each ***M(w)*** contains at least 3 elements and field F contains more than two elements. We refer to ***codim(ψ)=dim(Q)/N*** as codimension of an algebraic graph ***ψ***.

Studies of algebraic graphs with some restrictions on their cycles are motivated by the following 3 areas in Mathematics.

1. Investigations in the case of finite case are motivated by Extremal Graph Theory.

2. Flag transitive geometries over arbitrary fields are classical objects of Algebraic Geometry, they are incidence graphs i. e. simple graphs of binary relations defined over algebraic varieties over field ***F*** such that their edge sets are also algebraic varieties over ***F*.**

Rank two geometries are building bricks for geometries of higher rank. Their definitions are given in terms of girth and diameter.

For example classical projective plane is a graph of girth ***6*** and diameter ***3***. Its vertex set is a disjoint union of one dimensional

and two dimensional vector spaces of F^3. J. Tits defined generalised ***m***-gons as a bipartite graph of girth ***2m*** and diameter ***m*.**

Noteworthy that geometries of Chevalley groups $A_2(F)$, $B_2(F)$ and $G_2(F)$ are generalised ***m***-gons for ***m=3, 4*** and ***6***.

3. Studies of families ***G_i (F)*** of homogeneous algebraic graphs defined over the field ***F*** with well defined projective limits ***G(F)*** when ***n*** tends to infinity form an interesting direction of Algebraic Geometry. The cases when ***G(F)*** is a forest or a tree are especially important. Investigations of growth of order of maximal or minimal cycles in ***G_i(F)*** are naturally required in this cases

In probability theory **branching process** is a special stochastic process corresponding to random walk on regular forest ***For*** , i. e. simple graph without cycles with vertexes of the same degree of finite or infinite of cardinality **>2**.
The genealogy of single vertex is a tree.

The forest ***For*** itself is a deterministic part of branching process.

A possibility to define ***For*** by system of equations over some field of special commutative ring ***K*** i.e. as a projective limit of homogeneous algebraic graphs ***G_i*** ***, i=1,2, ...*** of increasing girth defined over ***K motivate special direction of Infinite Network Theory.***

Let us introduce some definitions of homogeneous algebraic graph theory

We refer to ***G*** as infinite algebraic graph over ***K*** if ***G*** is a projective limit for the family G_i $i=1,2,\ldots$ of ***k***-homogeneous algebraic graphs.

If ***G*** is a forest we say that the family G_i of ***k***-homogeneous graphs is an algebraic *forest approximation* over commutative ring ***K.***

Let g_i stands for the girth of G_i.

In the case $g_i \geq cn_i$, where n_i are dimensions of the vertex sets $V(G_i)$ of the graph G_i and ***c*** is some positive constant we use term ***algebraic forest approximation of large girth.***

Let ***G*** be a connected graph and d_i stands for the diameter *of* G_i

In the case $d_i \leq cn_i$, where n_i are dimensions of the vertex sets $V(G_i)$ of the graph G_i and c is some positive constant we use term ***algebraic G-approximation via small world graphs.***

The existence of tree approximations defined over arbitrary field is proven.

Noteworthy that the algebraic forest approximation over finite field is a family of finite graphs of large girth in the sense of P. Erdős'.

The first example of the family of graphs of large girth over arbitrary field was introduced in [1998] where was stated that graphs $D(n,K)$ over arbitrary infinite integrity domain have girth $\geq 2[(n+5)/2]$. This fact was proven in [Ustimenko, Journal of Math Sci, 2007]. A bit more compact prove without usage of terminology of linguistic dynamic systems theory is given in [IACR e-print archive].

Noteworthy that together with $D(n, K)$, $n=2,3,...$ one can consider another families $D(n, K[x_1, x_2, ..., x_m])$ for

each parameter ***m.*** It opened a possibility to use extremal properties of these graphs in the **Theory of Symbolic Computations.**
In [T. Shaska , Ustimenko] it was proven that the girth of ***D(n, F)*** defined **over the field *F* of characteristic zero** equals 2*[*(***n+5)/2].***

Note that small world approximation of infinite algebraic graph over finite field is a family of small world graphs in sense of [Bolloba's', Random Graphs].

The first small world approximation of infinite algebraic graph was presented in [Futorny, Ustimenko, Acta Applicandae Mathematicae, 2007], where projective limit of Wenger graph ***W(n, F)*** where ***F*** is an infinite field was investigated.

Other definitions of Homogeneous Algebraic Graph Theory are motivated by the following statement.

Theorem [T. Shaska, V. Ustimenko, Lin Alg and its Appl].

Let ***G*** be the homogeneous algebraic graph over a field ***F*** of girth ***g*** such that the dimension of a neighbourhood for each vertex is ***N, N*** **≥*1.*** Then ***codim(G) = dim(Q)/N*** **≥*[(g - 1)/2].***

We introduce ***v(g)*** as minimal value of ***codim(G)*** for homogeneous algebraic graph ***G*** of girth ***g.*** We refer

We refer

to ***v(g)*** as ***algebraic rank*** of girth ***g.***

Corollary.

v(g) ≥ [(g-1)/2]

We refer to graph ***G*** of girth ***g*** and ***codim(G)=v(g)*** as *algebraic cage.*

In the case of graph ***G*** of girth **g** and ***codim(G)=[(g-1)/2]*** we say that ***G*** is *algebraic Moore graph.*

Theorem 1.

Let $v(g)$ be the minimal codimension of homogeneous algebraic graph of even girth $g=2k+2,\ k \geq 6$.
Then

$$k \leq v(g) \leq (3k-3+e)/2$$ where $e = 0$ if k is odd, and $e = 1$ if k is even.

(graphs CD(n,F))

Let F be a field $F \neq F_2$. We introduce ${}^{F}v(g)$ as minimal ***codim(G)*** for

algebraic graph G over the field F with girth g. •

If g , $g \geq 6$ is even then ${}^{F}v(g)$ is at least $(g-2)/2$, for each field F, $F \neq F_2$.

The upper bound for ${}^{F}v(g)$ can be heavy dependable from the choice of field.

For each even ***m*** we introduce ***t(m)*** as minimal codimension of homogeneous algebraic graph without cycle C_m. Noteworthy that

$t(m) \le v(m+2)$.

The following conjecture is motivated by Even Circuit Theorem.

CONJECTURE 1. ***The inequality*** $t(m) \ge m/2$ ***holds.***

We refer to the written above inequality as Even Circuit Inequality (ECI).

and to homogeneous algebraic graphs without C_m of codimension $= m/2$

as algebraic Turan graphs.

THEOREM 2.

Let ***m=4, 6, 8,10.*** There are algebraic Turan graphs without cycles

C_4, C_6, C_8 , C_{10} of codimensions 3, 4, 5 and 6 respectively.

(regular generalised **m**-gons for ***m=3, 4 , 6*** and graph ***A(4, 4)).***

Theorem 3.

Let ***F*** be a fields with at least 3 elements

Let $^{F}t(m)$ be the minimal codimension of homogeneous algebraic graph over ***F*** without cycles of length $m=2k$, $k \geq 6$.

Then

$^{m}t(m) \leq (3k-3+e)/2$ where $e = 0$ if k is odd, and $e = 1$ if k is even.

(*graphs CD(n, F))*

THEOREM.

$6 \le v(14) \le {}^{F}v(14) \le 7$

It follows from the fact that the girth of algebraic graph $A(7, F_4)$ equals to 14.

CONJECTURE.

$v(14) = 7$ *and* $A(7; F_4)$ *is an algebraic cage*

1.3 ANOTHER OPTIMISATION PROBLEM FOR FINITE OR HOMOGENEOUS ALGEBRAIC GRAPHS

Problems on evaluation of girth and diameter of ***k-***regular simple graph with ***k ≥3*** are well known. Additionally we consider following optimization ‘’minimax’’ problems for graphs.

(1) Investigate cycle indicator ***h(v)*** of the vertex ***v*** of the ***k-***regular graph ***G,*** i. e. the minimal length of cycle through this vertex ***v.***

(2) Find the cycle indicator ***h(G)*** of the graph which is maximal value of cycle indicators of vertexes of the graph.

As it instantly follows from the definitions $h(G) \geq g(G)$, where $g(G)$ stands for the girth of the graph, which is minimal size of a cycle of ***G***.

We say that family G_i, $i=1, 2, \ldots$ of increasing order v_i.is a family with large girth indicator if cycle indicator $h(i)$ of graph G_i are at least $c\log_{k-1}(v_i)$ for some independent positive constant c.

Similarly we say that family of homogeneous algebraic graphs G_i., $i=1,2,\ldots,n$ defined over the field F with increasing dimensions d_i of vertex sets $V(G_i)$ such that the neighbourhood of each vertex of G_i. has fixed dimension N independent from parameter i

is an algebraic family of graphs with large cycle indicator if cycle indicator $h(i)$ of graph G_i are at least cd_i for some positive constant i.

As it follows from definitions each family of graphs (or algebraic graphs) of large girth is a family of graphs (algebraic graphs) with large circle indicator. So quite many examples of such families are known (see [LUW] and further references)

We introduce a family of algebraic graphs over arbitrary field ***F*** with the large circle indicator with constant $c=2$. If $F=F_q$ this is the family of finite graphs with large circle indicator and constant $c=2\log_q(q-1)$.

Let ***G*** be ***k***-homogeneous algebraic graph over field ***F*** with the vertex set ***V(G)*** and with the girth indicator ***h***

Let ${}^{F}u(h)$, $h \geq 6$ be the minimal codimension for variety of such graphs defined over the field ***F***. We consider $u(h)$ as minimal value of ${}^{F}u(h)$ via all fields ***F*** with at least 3 elements. Noteworthy that ${}^{F}u(h) \geq u(h)$.

Theorem 4.

For each even **h, h≥6** *the inequality* **u(h)≤(h-2)/2** *holds.*

Justification of this statement can be given via evaluation of cycle indicator of graphs **A(n, F)** *defined below* .

The evaluations of $^{F}u(h)$ *for special fields F also can be given* .

CONJECTURE 2.

The bound of Theorem 4 is sharp, i. e. **u(h)=(h-2)/2** *for even* **h, h≥6.**

Let ***K*** be a commutative ring .

We define $A(n, K)$ as bipartite graph with the point set $P=K^n$ and line set $L=K^n$ (two copies of a Cartesian power of K are used). We will use brackets and parenthesis to distinguish tuples from P and L.

So $(p)=(p_1, p_2, \ldots , p_n) \in P_n$ and $[l]=[l_1, l_2 , \ldots , l_n] \in L_n$.

The incidence relation $I=A(n,K)$ (or corresponding bipartite graph I) is given by condition $p\ I\ l$ if and only if the equations of the following kind hold.

$p_2 - l_2=l_1p_1,$

$p_3 - l_3= p_1 l_2,$

$p_4 - l_4 = l_1p_3,$

$p_5 - l_5 = p_1 l_4 ,$

$\ldots ,$

$p_n - l_n = p_1 l_{n-1}$ for odd n and $p_n - l_n = l_1 p_{n-1}$ for even n.

We can consider an infinite bipartite graph ***A(K)*** with points

(p_1, p_2 ,…, p_n ,…) and lines ***[l_1 , l_2 ,…,l_n , …].***

We proved that for each odd ***n*** girth indicator of ***A(n, K)*** is at least ***2n+2.***

GRAPHS *D(n, K)*

The following interpretation of a family of graphs ***D(n. K)*** in case of general commutative ring ***K*** is convenient for the computations. Let us use the same notations for points and lines as in previous case of graphs ***A(n, K).***

Points and lines are elements of two copies of the affine space over K. Point **(*p*)=(p_1, p_2, … , p_n)** is incident with the line ***[l]=[l_1, l_2 , … , l_n]*** if the following relations between their coordinates hold:

$p_2 - l_2 = l_1 p_1,$

$p_3 - l_3 = p_1 l_2,$

$p_4 - l_4 = l_1 p_3$,

…,

$l_i - p_i = p_1 l_{i-2}$ if i congruent to **2** or **3** modulo **4**,

$l_i - p_i = l_1 p_{i-2}$ if i congruent to **1** or **0** modulo **4.**

PROPOSITION (see [Archive] and further references).

*Let **K** be an integrity ring.*

*Then for each odd $n \geq 2$ a cycle indicator of graph **A(n, K)** is at least **2n+2.***

In the case when K coincides with F_q graphs $A(n, F_q)$ are q-regular, they form a family of graphs with large cycle indicator, appropriate constant c can be written as $2\log_q(q-1)$.

We prove inequality $h(A(n, K)) \geq 2n+2$ via computation of $h(v)$ for the vertex v (point or line) given by the tuple $(0, 0, \ldots , 0)$.

Computer simulation indicates that if $n > 6$ then cycle indicators of 0-point and 0-line are different, one of them is $2n+2$ but other is $2n$. It means that graphs are not vertex transitive, their girth differs from the cycle indicator in investigated via computer simulation cases.

CONJECTURE 3.

*Let **F** be a field with at least 3 elements. Then for each odd **n ≥3**, a cycle indicator of graph **A(n, F)** is **2n+2.***

The following statement supports Conjecture 3.

Theorem 5.

*Let **k , k>3** be odd number.*

*There are infinitely many fields **F** such that girth indicator of **A(k, F)** is **2k+2.***

•

Theorem 4 instantly follows from Theorem 5.

We refer to homogeneous algebraic graphs over field *F* with even cycle indicator *h* of codimension *(h-2)/2* *as cyclonic graphs*.

Class of cyclonic graphs contains

geometries of Chevalley groups $A_2(F)$, $B_2(F)$, $G_2(F)$ (known flag transitive generalised m-gons for $m=3$, 4, and 6 in the case of arbitrary field F). Other examples give some representatives of

known family of graphs $A(n, F)$ defined above. Noteworthy that in the cases of generalised polygons their girth coincides with the cyclic indicator,

but in well investigated via computer simulation cases of graphs $A(n, F_q)$, $q=3$, $q=4$ and $q=5$ and odd parameters n from the intervals [7,27], *[7, 15] and [7,13] these parameters are distinct. Computer simulation was conducted by Prof. G. Erskine.*

We define Cyclic gap of graph G as the difference between cycle indicator Cind(G) and Girth(G). Investigation of cyclic gaps of A(n, F) is an interesting open problem of Algebraic Geometry.

The proof of Theorem 5 uses the investigation of integrity domains K for which girth of $D(2k+1, K)$ is $2k+6$.

Theorem 4. 1. (Z. Füredi, F. Lazebnik, A. Seress, V.A. Ustimenko,

A.J. Woldar)r,)

*Let **k** be odd, and **q** be any prime power in the arithmetic progression* **{1 + *n*(*k* + 5)/2},**
n **= 1, 2, . ..** *Then the girth of* $D(k, F_q)$ *is* $k + 5$.

Theorem 4.2 (T. Shaska, V. Ustimenko). *Let **k** be odd, and **P** be the arithmetic progression* **P = {1 + *n*(*k* + 5)/2}, *n* = 1, 2, . . .** *Then*
(i) *for each integrity ring F of prime characteristic* $p \in P$ or ***0*** *the girth of the graph* **D(k, F)** *is* $k + 5$;
(ii) *there is an integer function* ***n(k)*** *such that for each commutative integrity domain **K** with unity such that* **char(K) ≥n(k)** *the girth of the graph* **D(n, K)** *is* $k + 5$.

Noteworthy that ***$D(m, F_q)$*** is induced subgraph of ***$D(m, F_q^s)$*** and the following statement instantly follows from Theorem 4.1.

Corollary 1.

Let parameters m and q satisfy condition of Theorem 4. 1 and s≥ 2.
Then the girth of *$D(m, F_q^s)$* is *m+5.*

The usage of this corollary allows us to formulate the following statement on existence of infinitely many finite cyclonic graphs

Theorem 4.3.

Let ***k*** *be odd, and* ***q*** *be any prime power in the arithmetic progression* **{1 + *n*(*k* + 5)/2}, *n* = *1, 2, …*** *and* ***s*** is *an integer* ***≥1.*** ***Then cycle indicator of A(r, F_{q^s}), r=(k+3)/2*** is **2*r*+2.**

The following statement on the existence of other examples of cyclonic graphs can be deduced from the Theorem 4.2.

Theorem 4. 4. *Let* ***k*** *be odd, and* ***P*** *be the arithmetic progression* ***P*** **= {1 + *n*(*k* + 5)/2}, *n* = 1, 2, . . .** *Then*

(i) *for each integrity ring F of prime characteristic* ***p*** **∈** ***P*** *or* ***0*** *the cycle indicator of* ***A(s, F)*** is **2*s*+2** *where* ***s=(k+3)/2***.

(ii) *there is an integer function* ***n(k)*** *such that for each* *commutative integrity ring* ***K*** *with unity such that* ***char(K) ≥n(k)*** *the girth indicator of the graph* ***A(s, F)*** *is* ***2s+2.***

Theorem 5 follows instantly from Theorem 4.3. or Theorem 4.4.

On diameter and girth of homogeneous algebraic graphs (work in progress)

LEMMA

Let ***G*** be the homogeneous algebraic graph over a field ***F*** of diameter ***d*** such that the dimension of a neighbourhood for each vertex is ***N, N ≥1.*** Then

d ≥ codim(G) = dim(V)/N. (1)

For the case of bipartite graphs or graphs without even cycles we can prove that the stronger lower bound on the diameter holds. It follows.

$d \geq dim(V/N)+1$. (2)

It can be justified via breadth-first search tree starting from the midpoint of a single edge.

We refer to bipartite homogeneous algebraic graph ***G*** as ***algebraic Tutte's graph if*** its codimension ***codim(G)*** *is* diameter ***d(G)*** minus ***1.***

In the case of algebraic Tutte's graphs which are also

algebraic Moore graphs we use term *Tits graphs.*

It is clear that in the case of even girth diameter of Tits graph is a half of the girth.

Obvious examples of Tits graphs are *generalised* ***m****-gons* with ***m=3, 4, 6*** *which* are geometries of Chevalley groups $A_2(F)$, $B_2(F)$ and $G_2(F)$ over arbitrary field ***F.*** They have codimensions ***2, 3*** and ***5*** and girth ***6, 8, 12*** respectively.

CONJECTURE 1. Tits graphs exists only in cases of codimensions ***2, 3*** and ***5.***

(in a spirit of Feit-Higman Theorem).

CONJECTURE 2. Algebraic Moore graphs exists only in cases of codimensions ***2,3, 4*** and ***5.***

We introduce some integer function in terms of algebraic geometry

which are extremely hard for computation in a following way,

g(d) is maximal girth of algebraic homogeneous graph of diameter d, $d \geq 3$

d(g) is minimal diameter of algebraic homogeneous graphs of girth ***g, g ≥4.***

It is easy to see that ***d(6)=3, d(8)=4, d(12)=6*** and

g(3)=6, g(4)=8, g(6)=12.

From the result below about girth and diameter of ***A(4, 4***) we get that ***d(10) ≤ 8*** and ***g(8)≥10.***

Recall that we introduce ***v(n)*** as minimal codimension of existing algebraic *homogeneous graph of girth* ***n.*** So ***v(n)≥[(n-1)/2] .***

We can investigate ***w(d)*** which is a maximal codimension of existing bipartite homogeneous algebraic graph of diameter ***w(d)≤ d-1.***

BRIDGE TO GROUPS OF TRANSFORMATIONS OF AFFINE SPACE K^n.

Linguistic graphs and symbolic computations.

Let K be a commutative ring. We refer to an incidence structure with a point set $P=P_{s,m}=K^{s+m}$ and a line set $L=L_{r,m}=K^{r+m}$ as linguistic incidence structure $I_m(K)$ of type (s, r, m) if point $x=(x_1, x_2, \ldots, x_s, x_{s+1}, x_{s+2}, \ldots, x_{s+m})$ is incident to line $y=[y_1, y_2, \ldots, y_r, y_{r+1}, y_{r+2}, \ldots, \ldots, y_{r+m}]$ if and only if the following relations hold

$a_1 x_{s+1} + b_1 y_{r+1} = f_1(x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_r),$

$a_2 x_{s+2} + b_2 y_{r+2} = f_2(x_1, x_2, \ldots, x_s, x_{s+1}, y_1, y_2, \ldots, y_r, y_{r+1}),$

…,

$a_m x_{s+m} + b_m y_{r+m} = f_m(x_1, x_2, \ldots, x_s, x_{s+1}, \ldots, x_{s+m}, y_1, y_2, \ldots, y_r, y_{r+1}, \ldots, y_{r+m}),$

where a_j and b_j, $j=1,2, \ldots, m$ are not zero divisors, and f_j are multivariate polynomials with coefficients from K. Brackets and parenthesis allow us to distinguish points from lines. The colour $\dot{\rho}(x)=\dot{\rho}((x))$ $(\dot{\rho}(y)=\dot{\rho}([y]))$ of point (x) (line $[y])$ is defined as projection of an element (x) (respectively $[y])$ from a free module on its initial s (relatively r) coordinates. As it follows from the definition of linguistic incidence structure $I_m(K)$ for each vertex of its incidence graph $\Gamma(m, K)$ there exists the unique neighbour of a chosen colour.

For each $b \in K^r$ and $p=(p_1, p_2, \ldots, p_{s+m})$ there is the unique neighbour of the point $[l]=N_b(p)$ with the colour b. Similarly, for each $c \in K^s$ and line $l=[l_1, l_2, \ldots, l_{r+m}]$ there is the unique neighbour of the line $(p)= N_c([l])$ with the colour c. We refer to operator of taking the neighbour of vertex accordingly chosen colour as *neighbourhood operator.*

Noteworthy, that the path in $v_0, v_1, \ldots, v_k$ the linguistic graph I_m is determined by starting vertex v_0 and colours of vertexes $v_1, v_2, \ldots, v_k$. We can take commutative ring $R=K[y_1,y_2,\ldots,y_l]$ and consider graph $I_m(K)$ together with infinite graph $I_m(R) = I_m(K[y_1, y_2, \ldots, y_l])$ defined by the same polynomials f_i, $i=1, 2, \ldots, m$ with coefficients from K but with partition sets R^{s+m} and R^{r+m}.

Assume that $l=m+s$. We can consider the path in $I_m(R)$ of length $2k$ in with starting point $(y_1, y_2, \ldots, y_s, y_{s+1}, y_{s+2}, \ldots, y_{s+m})$ and consecutive colours $G_1, H_1, G_2, H_2, \ldots, G_k, H_k$ such that $G_i \in K[x_1,x_2, \ldots, x_s]^s$ and $H_1 \in K[x_1, x_2, \ldots, x_s]^r$. The last vertex of this path will be a point (p) with consecutive coordinates $h_1, h_2, \ldots, h_s, f_{s+1}, f_{s+2}, \ldots, f_{s+m}$ where $f_1, f_2, \ldots, f_{s+m}$ are elements of $K[x_1, x_2, \ldots, x_s, x_{s+1}, x_{s+2}, \ldots, x_{s+m}]^s$

We define *passage transformation* $^{\Gamma(m,K)}Pas(G_1, G_2, \ldots, G_k, H_1, H_2, \ldots, H_k)$ of K^{r+s} (space of points) with symbolic colours $G_1, H_2, \ldots, G_k, H_k$ via multivariate rule $y_1 \rightarrow h_1(y_1, y_2, \ldots, y_s)$, $y_2 \rightarrow h_2(y_1, y_2, \ldots, y_s), \ldots, y_s \rightarrow h_s(y_1, y_2, \ldots, y_s)$, y_{s+1}

$\to f_{s+1}(y_1, y_2, \ldots, y_{s+m}), y_{s+2} \to f_{s+2}(y_1, y_2, \ldots, y_{m+s}), \ldots, y_{s+m} \to f_{s+m}(y_1, y_2, \ldots, y_{m+s})$. It is easy to see that this transformation is bijective if the map $y_i \to h_i(y_1, y_2, \ldots, y_s)$, $i=1, 2, \ldots, s$ is bijective on K^s.

We are searching for families of linguistic graphs $\Gamma(m, K)$ of type (r, s, m) with constant parameters r and s and $m=1, 2, \ldots$ such that standard forms of nonlinear transformations $F_m = {}^{\Gamma(m,K)}Pas(G_1, G_2, \ldots, G_k, H_1, H_2, \ldots, H_k)$, $k=O(m)$ where symbolic colours are taken from a special class of tuples have restricted degrees (2 or 3).

In the case of $A(n,K)$ and $D(n, K)$ of type $(1, 1, n-1)$, H_i, G_i of kind

$y+a_i$, $y+b_i$ transformations of kind F_m form ***a large group of cubic multivariate transformations*** of K^n ***of degree 3*** which is an interesting object of Algebraic Geometry.

Why?

Let ***K*** a field. Then the composition of two nonlinear maps ***g:K→K*** and ***h:K→K*** of degrees ***s*** and ***r*** will have degree ***r•s.***

In the case of ***K***^{n}***, n≥2*** the composition of two nonlinear map ***g*** and ***h*** will have degree ***max(r, s) ≤ d ≤ rs. For waste majority cases it will be rs.***

So constructions of new stable group of transformations of K^{n}***, i. e. groups formed by maps with maximal constant degree s is an interesting problem of Algebraic Group Theory.***

THANK YOU FOR YOUR ATTENTION