The feasibility of families of graphs characterised by forbidden induced subgraphs.

Christina Zarb (joint work with Yair Caro, Matthew Cassar and Josef Lauri)

September, 2024

The Feasibility Problem

Definition

Let $\mathcal{F}$ be an infinite family of graphs. Then $\mathcal{F}$ is called *feasible* if for every $n \geq 1$, $0 \leq m \leq \binom{n}{2}$, there is as graph $G \in \mathcal{F}$ having exactly n vertices and m edges.

If $\mathcal{F}$ is not feasible, the following sets and parameters are of interest:

- $FP(\mathcal{F}) = \{(n, m) : \exists$ a graph $G \in \mathcal{F}$ having exactly n vertices and m edges$\}$
- $\overline{FP}(\mathcal{F}) = \{(n, m) :$ no member of $\mathcal{F}$ has precisely n vertices and m edges$\}$
- $f(n, \mathcal{F}) = \min\{m : (n, m)$ is not a feasible pair for the familiy $\mathcal{F}\}$
- $F(n, \mathcal{F}) = \max\{m : (n, m)$ is not a feasible pair for the family $\mathcal{F}\}$

Examples

The Family $\mathcal{F}$ of Connected Graphs

Every connected graph on n vertices must have at least $n-1$ edges, so in this case $\mathcal{F}$ is not feasible.

- $FP(\mathcal{F}) = \{(n,m) : m \geq n-1\}$
- $\overline{FP}(\mathcal{F}) = \{(n,m) : m < n-1\}$
- $f(n,\mathcal{F}) = 0$
- $F(n,\mathcal{F}) = n-2$

The Family $\mathcal{F}$ of Planar Graphs

Here, based on the known fact that a maximal planar graph can have at most $3n-6$ edges for $n \geq 3$, we have

- $FP(\mathcal{F}) = \{(n,m) : m \leq 3n-6\}$
- $\overline{FP}(\mathcal{F}) = \{(n,m) : m > 3n-6\}$
- $f(n,\mathcal{F}) = 3n-5$
- $F(n,\mathcal{F}) = \binom{n}{2}$

Examples

The Family $\mathcal{F}$ of Triangle-Free Graphs

Based on a Theorem by Mantel which states that a triangle-free graph on n vertices has at most $\lfloor \frac{n^2}{4} \rfloor$ edges, then $\mathcal{F}$ is not feasible, and

- $FP(\mathcal{F}) = \{(n, m) : m \leq \lfloor \frac{n^2}{4} \rfloor\}$
- $\overline{FP}(\mathcal{F}) = \{(n, m) : m > \lfloor \frac{n^2}{4} \rfloor\}$
- $f(n, \mathcal{F}) = \lfloor \frac{n^2}{4} \rfloor + 1$
- $F(n, \mathcal{F}) = \binom{n}{2}$

Feasible Families

On the other hand, chordal graphs (a graph is chordal if all cycles on four or more vertices have a chord) and claw-free graphs are examples of feasible families.

Forbidden Subgraphs

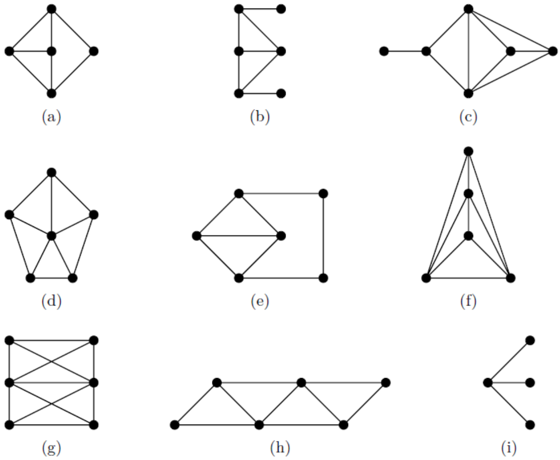
Definition

We say that a graph G is induced H-free or just H-free (when there is no ambiguity) if G has no induced copy of H. Generalising the definition, let $\mathcal{H}$ be a family of graphs. A graph G is said to be induced $\mathcal{H}$-free if no induced subgraph of G is a copy of a member of $\mathcal{H}$.

Several families of graphs can be characterised as induced $\mathcal{H}$-free. We have just mentioned triangle-free and claw-free graphs, and chordal graphs which are graphs in which induced cycles of length four or more are forbidden, that is chordal graphs are $\mathcal{H}$-free, where $\mathcal{H} = \{C_k : k \geq 4\}$.

Line Graphs

A very important family is the family of line graphs which is characterised as $\mathcal{H}$-free where $\mathcal{H}$ is the set of nine forbidden Beineke graphs shown below.



Line Graphs

Clearly, the family of line graphs is not feasible — the pair (5,9) is already an example on a non-feasible pair since the only graph on five vertices and nine edges is the graph $K_5 \backslash K_2$ which is one of the nine forbidden subgraphs.

The feasibility of the family of line graphs is discussed in the paper "The feasibility problem for line graphs" [FLG], which is the inspiration behind the work presented here.

G-free Graphs

We first consider the feasibility of induced G-free graphs and prove the following Theorem.

Theorem

Let G be a graph — the family $\mathcal{F}(G)$ of all induced G-free graphs is feasible if and only if G is not a member of

$$TNF = \{K_k, K_k \backslash K_2, \overline{K_k}, \overline{K_k \backslash K_2}\}$$

for $k \geq 2$.

In other words, if G is not a member of TNF, then for every pair (n, m), $n \geq 1$, $0 \leq m \leq \binom{n}{2}$, there is an induced G-free graph with exactly n vertices and m edges.

Basic properties

We first state some basic results about feasiblility in general.

Proposition

Let $\mathcal{F}$ and $\mathcal{H}$ be two families of graphs such that $\mathcal{H} \subset \mathcal{F}$. Then

1. *If $\mathcal{H}$ is a feasible family then $\mathcal{F}$ is feasible family.*
2. *If $\mathcal{F}$ is not a feasible family then $\mathcal{H}$ is not feasible family.*

Proposition

1. *Let $\mathcal{F}$ be a family of graphs and $\overline{\mathcal{F}} = \{\overline{G} : G \in \mathcal{F}\}$. Then $\mathcal{F}$ is feasible if and only if $\overline{\mathcal{F}}$ is feasible.*
2. *Let $\mathcal{F}(G)$ be the family of all induced G-free graphs . Then $\overline{\mathcal{F}} = \mathcal{F}(\overline{G})$, the family of all induced $\overline{G}$-free graphs, is feasible if and only if $\mathcal{F}(G)$ is feasible.*

The UEP

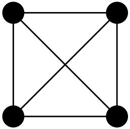
The Process

The Universal Elimination Process (UEP), introduced in FLG, is a method which is used to delete edges systematically from a complete graph. We start with K_n and order the vertices $v_1, \ldots, v_n$. We now delete at each step an edge incident with v_1 until v_1 is isolated. We then repeat the process of step by step deletion of the edges incident with v_2, and continue until we reach the empty graph on n vertices. Along the process, for any pair (n, m), $0 \leq m \leq \binom{n}{2}$, we have a graph G with n vertices and m edges.

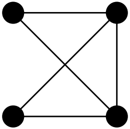
Lemma

The maximal induced subgraphs of K_n obtained when applying UEP on K_n are of the form $H(p, q, r) = (K_p \backslash K_{1,q}) \cup rK_1$, $p - 1 \geq q \geq 0$ and $p + r = n$.

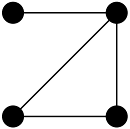
Applying UEP to K_4



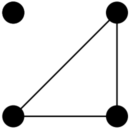
H(4,0,0)



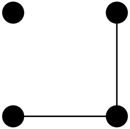
H(4,1,0)



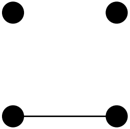
H(4,2,0)



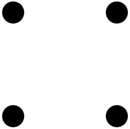
H(3,0,1)



H(3,1,1)



H(2,0,2)



H(0,0,4)

Examples of Feasibility of H-free families using the UEP

Corollary

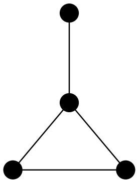
The following families of graphs obtained by applying the UEP are feasible:

1. *induced $K_{1,r}$-free for $r \geq 3$, where $K_{1,r}$ is the star with r leaves.*
2. *induced P_r-free for $r \geq 3$, where P_r is the path on r edges.*
3. *induced rK_2-free for $r \geq 2$ where rK_2 is the union of r disjoint edges.*

The UEP can be used to show that any family $\mathcal{F}(G)$ of G-free graphs is feasible when $G \neq H(p, q, r)$. The above are three such examples.

$\{K_3, K_2\}$-elimination

In the case of a family $\mathcal{F}(G)$ of induced G-free graphs when G is of the form $H(p, q, r)$, a different edge elimination process is required to investigate feasibility. In FLG, the family of paw-free graphs is proved to be feasible, where the paw graph is isomorphic to $H(4, 2, 0)$, using a different edge elimination technique, which we now develop and extend.



$H(4, 2, 0)$ - the Paw graph

$\{K_3, K_2\}$-elimination

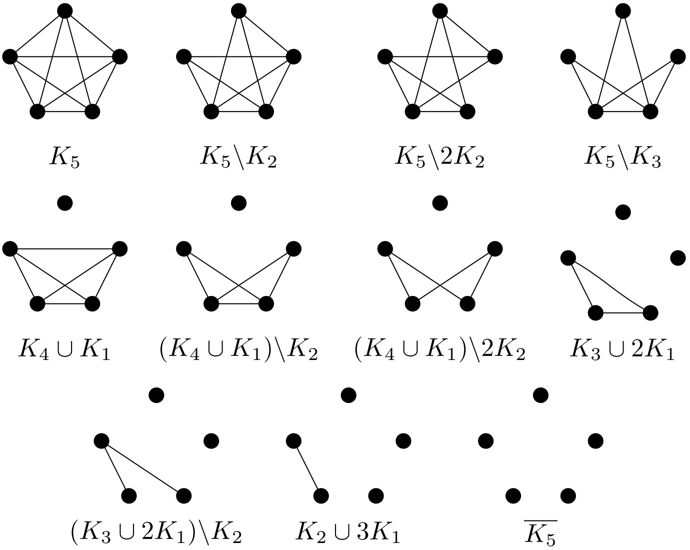
We first state a simple Lemma:

Lemma

For $n \geq 2$ and $0 \leq t \leq n-2$, there are integers $x, y \geq 0$ such that $3x + y = t$ and $xK_3 \cup yK_2$ is a subgraph of K_n.

The $\{K_3, K_2\}$-elimination process is described as follows: starting from K_n, for every $0 \leq t \leq n-2$, delete edges in the form $xK_3 \cup yK_2$ such that $3x + y = t$. Once this is done , we have covered all the range $[\binom{n-1}{2} + 1, \ldots, \binom{n}{2}]$. Consider now $K_{n-1} \cup K_1$ (obtained by deleting a star $K_{1,n-1}$) and apply the $\{K_3, K_2\}$-elimination process on K_{n-1} and continue until all edges are deleted. Once again observe that this process covers all possible numbers of edges in the range $[0, \binom{n}{2}]$.

Applying $\{K_3, K_2\}$-elimination to K_5



$\{K_3, K_2\}$-elimination

The graphs obtained through this elimination process are of the form $Q(p, r, x, y) = (K_p \backslash \{xK_3 \cup yK_2\}) \cup \overline{K_r}$ for $p, r, x, y \geq 0$ and $0 \leq 3x + y \leq p$ and $p + r = n$.

Lemma

The graphs obtained through the (K_3, K_2)-elimination process are induced $H(p, q, r)$-free for $p \geq 4$, $r \geq 0$, $2 \leq q \leq p - 2$. In particular the family $\mathcal{F}$ of all $H(p, q, r)$-free graphs is feasible whenever $p \geq 4$, $r \geq 0$, $2 \leq q \leq p - 2$.

Concluding the Proof

We shall now complete the proof. We have to consider the following remaining cases for $H(p, q, r)$:

1. the case $q = 0$
2. the case $p = 2$
3. the case $p = 3$
4. the case $q = 1$ and $q = p - 1$ when $p \geq 4$

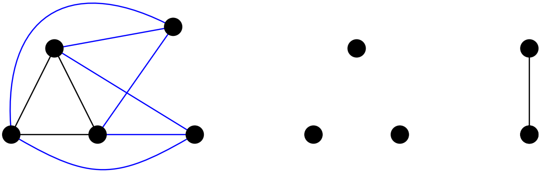
Each of these cases can be dealt with individually.

The case $q = 0$

Definition

Let $S(p, r)$ denote the graph $K_p + \overline{K_r}$, namely a clique K_p and an independent set $\overline{K_r}$ and all edges between the vertices in K_p and $\overline{K_r}$.

Observe that $S(p, r) = \overline{H(r, 0, p)}$, the complement of $H(r, 0, p)$.



S(3,2)

$H(2, 0, 3) = \overline{S(3, 2)}$

The case $q = 0$

Lemma

The feasibility of $\mathcal{F}(S(p, r))$:

1. For $p = 0$ or $r = 0$, $\mathcal{F}(S(p, r))$ is not feasible.
2. For $p \geq 1$, $r \in \{1, 2\}$, $\mathcal{F}(S(p, r))$ is not feasible.
3. For $p \geq 1$, $r \geq 3$, $\mathcal{F}(S(p, r))$ is feasible.

Corollary

The feasibility of $\mathcal{F}(H(p, 0, r))$.

1. For $p = 0$ or $r = 0$, $\mathcal{F}(H(p, 0, r))$ is not feasible.
2. For $p \in \{1, 2\}$ and $r \geq 1$, $\mathcal{F}(H(p, 0, r))$ is not feasible.
3. For $p \geq 3$, $r \geq 1$, $\mathcal{F}(H(p, 0, r))$ is feasible.

The cases $p = 2$ and $p = 3$

The case $p = 2$

Observe that $p = 2$ gives either $H(2, 0, r) = K_2 \cup \overline{K_r}$, or $H(2, 1, r) = \overline{K_{r+1}}$ which belong to the family TNF.

The case $p = 3$

Observe that $p = 3$ gives $H(3, 0, r)$, $H(3, 1, r)$, $H(3, 2, r) = H(2, 0, r + 1)$. We consider each of these graphs:

1. If $G = H(3, 0, r)$ then if $r = 0$, $G = K_3$ belongs to TNF and $\mathcal{F}(G)$ is not feasible, while if $r \geq 1$ then $\mathcal{F}(G)$ is feasible.
2. If $G = H(3, 2, r) = H(2, 0, r + 1)$, $\mathcal{F}(G)$ is not feasible.
3. If $G = H(3, 1, r)$ then if $r = 0$, $G = K_3 \backslash K_2$ which is a member of TNF and hence not feasible. If $r \geq 1$ then $G = K_3 \backslash K_2 \cup \overline{K_r}$. When $r = 1$, $G = \overline{H(4, 2, 0)}$ which is feasible and hence $\mathcal{F}(G)$ is feasible. For $r \geq 2$, $G = \overline{K_{p+r} \backslash K_{1,2}}$ which is feasible by $\{K_3, K_2\}$-elimination since $p + r \geq 5$ and we can delete $2K_2$.

The case $p \geq 4$

The case $q = p - 1$

For $p \geq 4$ and $q = p - 1$, $H(p, p-1, r)$ is feasible for $r \geq 0$.

The case $q = 1$

For $p \geq 4$ and $q = 1$, $H(p, 1, r)$ is feasible for $r \geq 1$ and not feasible for $r = 0$ since $H(p, 1, 0) = K_p \backslash K_2$, a member of TNF.

Hence we have proved that $\mathcal{F}(G)$ is feasible if and only if G is not one of the graphs K_k, $K_k \backslash K_2$, $\overline{K_k}$, $\overline{K_k \backslash K_2}$.

$\mathcal{H}$-free when $\mathcal{H}$ consists of two or more graphs

A natural question

Suppose G and H are graphs such that $\mathcal{F}(G)$ and $\mathcal{F}(H)$ are both feasible families. Is $\mathcal{F}(G, H)$, the family of all graphs which are simultaneously induced G-free and induced H-free, necessarily feasible ?

If both G and H are not $H(p, q, r)$ graphs then $\mathcal{F}(G, H)$ is feasible by UEP. This can be extended to more than two graphs and we give some examples.

Examples

Chordal graphs

Recall that chordal graphs are $\mathcal{H}$-free, where $\mathcal{H} = \{C_k : k \geq 4\}$. Cycles on four or more vertices are not of the form $H(p, q, r)$ and hence by UEP, chordal graphs are feasible.

Split Graphs

A split graph G is a graph whose vertex set can be partitioned into a clique and an independent set. It also has the property that G and $\overline{G}$ are both chordal and it can be characterised as $\mathcal{H}$-free, where $\mathcal{H} = \{2K_2, C_4, C_5\}$. These graphs are not of the form $H(p, q, r)$ so the UEP implies that the family of split graphs is feasible. Note that $H(p, q, r)$ are infact split graphs.
Also, since split graphs are a subfamily of chordal graphs, the feasibility of split graphs implies the feasibility of chordal graphs.

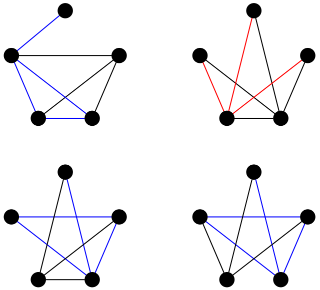
The Family $\mathcal{F}(Paw, Claw) = (H(4,2,0), K_{1,3})$

We consider the family $\mathcal{F}(Paw, Claw) = (H(4,2,0), K_{1,3})$. The family of paw-free graphs and the family of claw-free graphs are both feasible but

Theorem

$\mathcal{F}(Paw, Claw) = \mathcal{F}(H(4,2,0), K_{1,3})$ *is not a feasible family.*

The pair (5,7) forces an induced *Paw* or *Claw* as shown in the figure below.



A further consideration

Another interesting question is: since $\mathcal{F}(Paw, Claw)$ is not a feasible family, is $\mathcal{F}(Paw, K_{1,4})$ a feasible family or not? The Theorem below answers this question:

Theorem

$\mathcal{F}(Paw, K_{1,4})$ is a feasible family and so is $\mathcal{F}(Paw, K_{1,r})$ for $r \geq 5$.

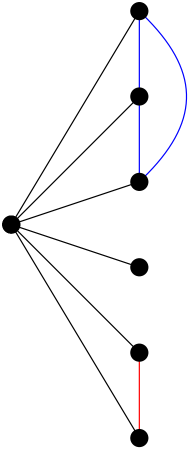
Sketch of Proof

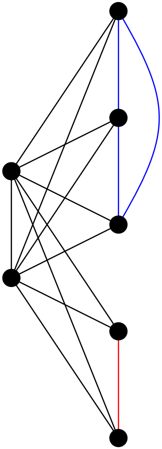
We consider the complementary family $\mathcal{F} = \mathcal{F}(P_3 \cup K_1, K_4 \cup K_1)$ and show that it is feasible. Then by Proposition 2, $\mathcal{F}(Paw, K_{1,4})$ is also feasible.

We use the split graphs $K_p + \overline{K_{n-p}}$ for $0 \leq p \leq n-3$ to construct graphs which cover the range $0 \leq m \leq \binom{n}{2} - 2$. In general, the split graph $K_p + \overline{K_{n-p}}$ has $\frac{p(n-1)+(n-p)p}{2}$ edges and we can pack, in the independent part $\overline{K_{n-p}}$ of order $n-p$, graphs with k edges of the form $aK_3 \cup bK_2 \cup cK_1$ with $n-p = 3a+2b+c$ and $k = 3a+b$ for $0 \leq k \leq n-p-2$, and cover the range of values of m from $\frac{p(n-1)+(n-p)p}{2}$ to $\frac{p(n-1)+(n-p)p}{2} + n - p - 2$.

- When $p = 0$ we start with the empty graph $\overline{K_n}$ and pack up to $n-2$ edges.
- When $p = 1$ we start with the graph $K_1 + \overline{K_{n-1}} = K_{1,n-1}$ and cover the range $n-1$ up to $n-1+n-3 = 2n-4$ edges.
- When $p = n-3$, we cover the range $\frac{n^2-n-6}{2} = \binom{n}{2} - 3$ up to $\binom{n}{2} - 3 + n - (n-3) - 2 = \binom{n}{2} - 2$.
- The final two values of m, which are $\binom{n}{2} - 1$ and $\binom{n}{2}$ are covered by the graph $K_n \backslash K_2$ (which is in fact $K_{n-2} + \overline{K}_2$), and K_n itself, both graphs being in $\mathcal{F}(P_3 \cup K_1, K_4 \cup K_1)$.

Examples for $n = 7$





$K_1 + \overline{K_6} = K_{1,6}$ with 6 edges
can add up to 4 edges

$K_2 + \overline{K_5}$ with 11 edges
can add up to 3 edges

The Feasibility of Line Graphs

In the context of line graphs, the feasibility of a pair (N, M) is closely related to the number-theoretic problem of representing non-negative integers by a sum of triangular numbers, which dates back to Gauss who proved that every non-negative integer n is representable by the sum of three triangular numbers. This is because it is well known that if G is a graph on $n(G) = n$ vertices and $e(G) = m$ edges with degree sequence $d_1 \leq \ldots \leq d_n$, then the line graph $L(G)$ has $n(L(G)) = m$ vertices and $e(L(G)) = \sum_{j=1}^{n} \binom{d_j}{2}$. (Harary)

The main distinction between the number-theoretic problem and the feasibility problem for line graphs of all graphs lies in the fact that in order to have a line graph $L(G)$ with N vertices and M edges, making (N, M) a feasible pair we require that $\sum_{j=1}^{s} d_j = 2N$ is obtained by a graphical degree sequence with a realizing/underlying graph G having $m = N$ edges (the number of vertices is not important) and the line graph $L(G)$ has $n(L(G)) = N$ while $\sum_{j=1}^{s} \binom{d_j}{2} = M$.

The Feasibility of Line Graphs

For the family of line graphs $\mathcal{F}$, $\overline{FP}(\mathcal{F})$ and hence $FP(\mathcal{F})$ are fully determined in the following Theorem:

Theorem (The Intervals Theorem)

For $N \geq 5$, all the values of M for which (N, M) is a non-feasible pair for the family of all line graphs, are exactly given by all integers M belonging to the following intervals:

$$\left[\binom{N-t}{2} + \binom{t+2}{2}, \ldots, \binom{N-t+1}{2} - 1\right] \textit{ for } 1 \leq t < \frac{-5+\sqrt{8N+17}}{2}.$$

Observe that if $\frac{-5+\sqrt{8N+17}}{2}$ is not an integer then the maximum value of $t = \left\lfloor \frac{-5+\sqrt{8N+17}}{2} \right\rfloor$ while if $\frac{-5+\sqrt{8N+17}}{2}$ is an integer then it is $t = \frac{-5+\sqrt{8N+17}}{2} - 1$.

Some numeric values

N	Non-feasible intervals
5	9
6	[13,14]
7	[18,20]
8	[24,27]
9	27,[31,35]
10	[34,35],[39,44]
20	135,[146,152],[157,170],[174,189]
30	[321,324],[340,350],[361,377],[384,405],[409,434]

The Feasibility of Line Graphs

The values of $f(n, \mathcal{F})$ is also determined in the Theorem below, while clearly, $F(n, \mathcal{F}) = \binom{n}{2} - 1$ for $n \geq 5$.

Theorem (The minimum non-feasible pair)

For $N \geq 2$, the minimum value of M which makes (N, M) a non-feasible pair, for the family of all line graphs, is $\binom{N-t}{2} + \binom{t+2}{2}$ where :

1. $t = \left\lfloor \frac{-5+\sqrt{8N+17}}{2} \right\rfloor$ *if* $\frac{-5+\sqrt{8N+17}}{2}$ *is not an integer.*
2. $t = \frac{-5+\sqrt{8N+17}}{2} - 1$ *if* $\frac{-5+\sqrt{8N+17}}{2}$ *is an integer.*

Values of $f(n, \mathcal{F})$

N	M
1	*
2	*
3	*
4	*
5	9
6	13
7	18
8	24
9	27
10	34

N	M
11	42
12	51
13	61
14	65
15	76
16	88
17	101
18	115
19	130
20	135

N	M
21	151
22	168
23	186
24	205
25	225
26	246
27	252
28	274
29	297
30	321

Publications

This presentation is based upon the following publications:

Y. Caro, J. Lauri, and C. Zarb.
The feasibility problem for line graphs.
Discrete Applied Mathematics, 324:167–180, 2023.

Y. Caro, M. Cassar,J. Lauri, and C. Zarb.
The Feasibility Problem – the family $\mathcal{F}(G)$ of all induced G-free graphs.
arXiv preprint, arXiv:2311.01082, 2023.

Thank you!